On the composition of the Euler function and the sum of divisors function

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Édition du 29 décembre 2006

Abstract

Let $H(n) = \dfrac{\sigma(\phi(n))}{\phi(\sigma(n))}$, where $\phi(n)$ is Euler's function and $\sigma(n)$ stands for the sum of the positive divisors of n. We obtain the maximal and minimal orders of $H(n)$ as well as its average order, and we also prove two density theorems. In particular, we answer a question raised by Golomb.

AMS Classification: 11A25, 11N37, 11N56
Key words: sum-of-divisors function, Euler's function

§1. Introduction

Let ϕ be Euler's function and let σ be the sum of the divisors function. The composition of the functions σ and ϕ has been the object of several studies; see for instance Mąkowski and Schinzel [9], Pomerance [11], Sándor [12], Ford [2], Luca and Pomerance [8]. In 1993, Golomb [3] investigated the difference $\sigma(\phi(n)) - \phi(\sigma(n))$ showing that it is both positive and negative infinitely often, and asked what is the proportion of each.

In this paper, we answer this question of Golomb and more, by studying the behavior of the quotient

$$H(n) := \frac{\sigma(\phi(n))}{\phi(\sigma(n))}.$$

In particular, we obtain the maximal and minimal orders of $H(n)$, its average order, and we also prove two density theorems.

Given any positive real number x we write $\log x$ for the maximum between the natural logarithm of x and 1. If k is a positive integer, we write $\log_k x$ for the k-th iteration of the function $\log x$. Throughout this paper, p,

q and r stand for prime numbers, while γ stands for Euler's constant. We also use $\pi(x)$ for the number of primes up to x and $\omega(n)$ for the number of distinct prime factors of n.

Acknowledgments. The first author was supported in part by a grant from NSERC. The second author was supported in part by Grants SEP-CONACyT 46755, PAPIIT IN104505 and a Guggenheim Fellowship. The authors wish to thank the referee for helpful remarks and suggestions which improved the quality of this paper.

§2. Main results

Theorem 1. *The maximal order of $H(n)$ is $e^{2\gamma}\log_2^2 n$, that is*

$$\limsup_{n\to\infty}\frac{H(n)}{\log_2^2 n}=e^{2\gamma}.$$

Theorem 2. *There exists a positive constant δ such that the minimal order of $H(n)$ is $\delta/\log_2 n$, that is*

$$\liminf_{n\to\infty} H(n)\log_2 n=\delta.$$

Moreover $\delta\in[(1/40)e^{-\gamma},2e^{-\gamma}]$.

Theorem 3. *As $x\to\infty$,*

$$\frac{1}{x}\sum_{n\le x}H(n)=c_0\,e^{2\gamma}\,\log_3^2 x+O\left(\log_3^{3/2} x\right),$$

where

$$\begin{aligned} c_0 &= \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\frac{\phi(n)}{\sigma(n)}\\ &= \prod_p\left(1-\frac{3}{p(p+1)}+\frac{1}{p^2(p+1)}+\frac{(p-1)^3}{p^2}\sum_{i=3}^{\infty}\frac{1}{p^i-1}\right)\approx 0.4578. \end{aligned}$$

Theorem 4. *For each number u, $0 \le u \le 1$, the asymptotic density of the set of numbers n with*

$$H(n) > ue^{2\gamma}\log_3^2 n$$

exists, and this density function is strictly decreasing, varies continuously with u, and is 0 when $u = 1$.

In particular, Theorem 4 shows that $\sigma(\phi(n)) - \phi(\sigma(n))$ is positive for most n, thus providing an answer to Golomb's question.

Theorem 5. *The set $\{H(1), H(2), H(3), \ldots\}$ is dense in $[0, +\infty)$.*

§3. Preliminary results

Theorem A (Heath-Brown). *Let k and a be coprime positive integers. Then there exists a prime number $p \equiv a \pmod{k}$ which satisfies $p = O(k^{11/2})$.*

Proof. See Heath-Brown [6].

Remark. It has been shown by Alford, Granville and Pomerance [1] that for most values of k, one can replace the constant $11/2$ by $12/5 + \varepsilon$ for any fixed $\varepsilon > 0$. It can also be shown that if GRH holds, then the constant $11/2$ can be replaced by $2 + \varepsilon$ for any fixed $\varepsilon > 0$.

Theorem B. (Pomerance) *There exists a constant $\kappa > 0$ such that, for all positive integers n,*

$$\frac{\sigma(\phi(n))}{n} > \kappa.$$

Proof. See Pomerance [11].

Remark. This statement relates to a long standing conjecture of Mąkowski and Schinzel [9], which asserts that $\dfrac{\sigma(\phi(n))}{n} \ge \dfrac{1}{2}$. Recently, Ford [2] has shown that $\kappa \ge \dfrac{1}{39.4}$. Note also that the conjectured minimum $\frac{1}{2}$ is attained when n is a product of the first Fermat primes, such as $n = 2$, 6, 30, 510, 131070 and 8589934590.

Lemma 1. (Mertens' Theorem) *The estimate*

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right)$$

holds for large values of x.

Proof. This result is due to Mertens [10].

Lemma 2. $\liminf_{n\to\infty}\frac{\phi(n)\log_2 n}{n}=e^{-\gamma}$.

Proof. This result, which follows essentially from Mertens' Theorem, was first obtained by Landau [7].

Lemma 3. $\limsup_{n\to\infty}\frac{\sigma(n)}{n\log_2 n}=e^{\gamma}$.

Proof. This result also follows from Mertens' Theorem and was first obtained by Gronwall [4].

Lemma 4. *There exists a positive constant* c_1 *such that for large real numbers* x*, both* $\phi(n)$ *and* $\sigma(n)$ *are divisible by all prime powers* $p^a < c_1\log_2 x/\log_3 x$ *for all positive integers* $n<x$ *with* $O(x/\log_3^2 x)$ *exceptions.*

Proof. The above result for the case of the function $\phi(n)$ is Lemma 2 in [8]. To prove the result for the function $\sigma(n)$, let m be an arbitrary positive integer and write

$$S(x,m)=\sum_{\substack{\log_2 x\le q\le x\\ m\mid(q+1)}}\frac{1}{q}.$$

From the Siegel-Walfisz Theorem (see Theorem 5, Chapter II.8 in Tenenbaum [13]) and partial summation, it follows that there exist positive numbers c_1 and x_0 such that the inequality

$$S(x,m)\ge\frac{c_1\log_2 x}{\phi(m)}$$

holds for $x>x_0$ and all $m\le\log x$. Let $g(x)=c_1\log_2 x/\log_3 x$. Using Brun's sieve, it follows that the set $\mathcal{N}_m$ of numbers $n\le x$ which have no prime factor $q > \log_2 x$ congruent to -1 modulo m satisfies

$$\#\mathcal{N}_m < c_2 x \prod_{\substack{\log_2 x < q < \log x \\ m|(q+1)}} \left(1 - \frac{1}{q}\right) \le c_2 x \exp(-S(x, m)),$$

for some positive constant c_2. Assuming now x_0 is chosen large enough so that $\log x > g(x)$ for all $x > x_0$, we get that if $m = p^a < g(x)$, then

$$\#\mathcal{N}_{p^a} < \frac{c_2 x}{\exp(S(x, p^a))} < \frac{c_2 x}{\exp(\log_3 x)} = \frac{c_2 x}{\log_2 x}.$$

Summing up the above inequalities over all the $O(g(x)/\log g(x))$ prime powers $p^a < g(x)$, we get that

$$\sum_{p^a < g(x)} \#\mathcal{N}_{p^a} \ll \frac{x g(x)}{\log_2 x \log g(x)} \ll \frac{x}{\log_3^2 x}.$$

Finally, let $\mathcal{M}$ be the set of all positive integers $n \le x$ such that n is divisible by the square of a prime $q \ge \log_2 x$. Then

$$\#\mathcal{M} \le \sum_{q \ge \log_2 x} \frac{x}{q^2} \ll \frac{x}{\log_2 x \log_3 x} \ll \frac{x}{\log_3^2 x},$$

where we used the fact that

(1)
$$\sum_{p > z} \frac{1}{p^2} \ll \frac{1}{z \log z}.$$

Note now that if $n \le x$ is such that p^a does not divide $\sigma(n)$ for some $p^a < g(x)$, then either n is in $\mathcal{M}$ or n is in

$$\bigcup_{p^a < g(x)} \mathcal{N}_{p^a},$$

and by the above estimates both these sets are of cardinality $O(x/\log_3^2 x)$, thereby completing the proof of Lemma 4.

Lemma 5. *Let x be a positive real number. Setting*

$$h_\phi(n) = \sum_{\substack{p|\phi(n) \\ p > \log_2 x}} \frac{1}{p} \qquad \text{and} \qquad h_\sigma(n) = \sum_{\substack{p|\sigma(n) \\ p > \log_2 x}} \frac{1}{p},$$

then

(2) $$\sum_{n\le x} h_\phi(n) \ll \frac{x}{\log_3 x} \quad \text{and} \quad \sum_{n\le x} h_\sigma(n) \ll \frac{x}{\log_3 x}.$$

Proof. Clearly we have

$$\sum_{\substack{n\le x\\ p\mid\phi(n)}} 1 \le \frac{x}{p^2} + \sum_{\substack{q\le x\\ p\mid q-1}} \frac{x}{q} \ll \frac{x}{p^2} + \frac{x\log_2 x}{p} \ll \frac{x\log_2 x}{p}.$$

It now follows that

$$\sum_{n\le x} h_\phi(n) = \sum_{p\le x}\frac{1}{p}\sum_{\substack{n\le x\\ p\mid\phi(n)}} 1 \ll x\log_2 x \sum_{p>\log_2 x}\frac{1}{p^2} \ll \frac{x}{\log_3 x},$$

where we used (1) with $z := \log_2 x$, thus establishing the first assertion in (2). We use a similar argument to establish the second assertion in (2). First of all, note that since $\omega(n) < \log x$ holds for all $n \le x$ provided x is large enough, it follows that

$$h_\sigma(n) \le \sum_{i\le \log x}\frac{1}{p_i} \ll \log_3 x,$$

where we used p_i to denote the i-th prime number. Let $\mathcal{N}_1$ be the set of all positive integers $n \le x$ such that there exists a prime $q > \log_3^2 x$ whose square divides n. Then, using (1),

$$\#\mathcal{N}_1 \le \sum_{q>\log_3^2 x}\frac{x}{q^2} \ll \frac{x}{\log_3^2 x\log_4 x}.$$

Hence,

(3) $$\sum_{n\in\mathcal{N}_1} h_\sigma(n) \ll \#\mathcal{N}_1 \log_3 x \ll \frac{x}{\log_3 x\log_4 x}.$$

Now let $\mathcal{N}_2$ be the set of those $n \le x$ which are not in $\mathcal{N}_1$ and which are divisible by a prime power q^a, with $a = \lfloor c_3 \log_4 x\rfloor + 2$, where $c_3 := 2/\log 2$. For a fixed prime number q, the number of such numbers n is $\le x/q^a$, and therefore

$$\#\mathcal{N}_2 \le \sum_{q\ge 2}\frac{x}{q^a} \le \frac{x}{2^a} + x\int_2^\infty \frac{dt}{t^a} \ll \frac{x}{2^a} \ll \frac{x}{\log_3^2 x},$$

which implies that

$$\sum_{n\in\mathcal{N}_2} h_\sigma(n) \ll \#\mathcal{N}_2 \log_3 x \ll \frac{x}{\log_3 x}. \tag{4}$$

Finally, let $\mathcal{N}_3$ be the set of positive integers $n \le x$ which do not belong to either $\mathcal{N}_1$ or $\mathcal{N}_2$. If $n \in \mathcal{N}_3$ and $q^{\alpha_q}||n$ with $\alpha_q > 1$, then $q < \log_3^2 x$ and $\alpha_q \ll \log_4 x$, so that $q^{\alpha_q} \le \exp(O(\log_4^2 x))$. Hence $\sigma(q^{\alpha_q}) < \exp(O(\log_4^2 x))$. In particular, for large x, we have that $\sigma(q^{\alpha_q}) < \log_2 x$. Hence, if $n \in \mathcal{N}_3$ and $p > \log_2 x$ is a prime dividing $\sigma(n)$, it follows that there exists a prime factor $q||n$ of n such that $p|(q+1)$. Now the same argument used for the function h_ϕ tells us that if $p > \log_2 x$ is a fixed prime, then

$$\sum_{\substack{p|\sigma(n)\\ n\in\mathcal{N}_3}} 1 \ll \sum_{\substack{q\le x\\ p|q+1}} \frac{x}{q} \ll \frac{x\log_2 x}{p}.$$

Therefore

$$\sum_{n\in\mathcal{N}_3} h_\sigma(n) \le \sum_{p>\log_2 x} \frac{1}{p} \sum_{\substack{p|\sigma(n)\\ n\in\mathcal{N}_3}} 1 \ll \sum_{p\ge\log_2 x} \frac{x\log_2 x}{p^2} \ll \frac{x}{\log_3 x}. \tag{5}$$

The second estimate (2) then follows from estimates (3), (4) and (5), and the proof of Lemma 5 is complete.

Lemma 6. *As $x \to \infty$,*

$$\sum_{n\le x} \frac{\phi(n)}{\sigma(n)} = c_0 x + O(x^{3/4}),$$

where c_0 is the constant appearing in the statement of Theorem 3.

Proof. Given any number s with $\Re(s) > 1$ and letting $\zeta(s)$ stand for the Riemann Zeta Function, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\phi(n)/\sigma(n)}{n^s} &= \prod_p \left(1 + \frac{\frac{p-1}{p+1}}{p^s} + \frac{\frac{p(p-1)}{p^2+p+1}}{p^{2s}} + \frac{\frac{p^2(p-1)}{p^3+p^2+p+1}}{p^{3s}} + \dots\right)\\ &= \zeta(s)\prod_p\left(1-\frac{1}{p^s}\right)\prod_p \left(1 + \frac{\frac{p-1}{p+1}}{p^s} + \frac{\frac{p(p-1)}{p^2+p+1}}{p^{2s}} + \frac{\frac{p^2(p-1)}{p^3+p^2+p+1}}{p^{3s}} + \dots\right)\end{aligned}$$

$$
\begin{aligned}
&= \zeta(s)\prod_p\left(1+\frac{\frac{p-1}{p+1}-1}{p^s}+\frac{\frac{p(p-1)}{p^2+p+1}-\frac{p-1}{p+1}}{p^{2s}}+\frac{\frac{p^2(p-1)}{p^3+p^2+p+1}-\frac{p(p-1)}{p^2+p+1}}{p^{3s}}+\dots\right)\\
&= \zeta(s)R(s),
\end{aligned}
$$

say. Expanding the product $R(s)$ into a Dirichlet series, say

$$
R(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},
$$

then it converges absolutely in the half-plane $\Re(s) \geq \frac{3}{4}$. Setting $b_n = \phi(n)/\sigma(n)$, we have $b_n = \sum_{d|n} a_d$, and therefore

$$
\begin{aligned}
\sum_{n\le x} b_n &= \sum_{n\le x}\sum_{d|n} a_d = \sum_{n\le x} a_d\left[\frac{x}{d}\right] = x\sum_{d\le x}\frac{a_d}{d}+O\left(\sum_{d\le x}|a_d|\right)\\
&= R(1)x + O\left(x\sum_{d>x}\frac{|a_d|}{d}\right)+O\left(\sum_{d\le x}|a_d|\right).
\end{aligned}
$$

Since

$$
\sum_{d\le x}|a_d| = \sum_{d\le x}\frac{|a_d|}{d^{3/4}}d^{3/4} = O(x^{3/4})
$$

and

$$
\sum_{d>x}\frac{|a_d|}{d} = \sum_{d>x}\frac{|a_d|}{d^{3/4}}\frac{1}{d^{1/4}} \le x^{-1/4}\sum_{d>x}\frac{|a_d|}{d^{3/4}} = O\left(x^{-1/4}\right),
$$

it follows that

$$
\sum_{n\le x}\frac{\phi(n)}{\sigma(n)} = R(1)x + O(x^{3/4}),
$$

which completes the proof of Lemma 6, since $R(1) = c_0$.

Lemma 7. *There exists a constant c_4 such that the set of positive integers $n \le x$ such that $\omega(\phi(n)) > c_4 \log_2^3 x$ contains at most $O(x/\log_2^2 x)$ elements. The same holds when the ϕ function is replaced by the σ function.*

Proof. First let $\mathcal{D}_1$ be the set of all $n \le x$ such that $k = \omega(n) > 3e\log_2 x$. A well-known result of Hardy and Ramanujan (see [5]) asserts that

$$
\#\{n \le x \ : \ \omega(n) = k\} \ll \frac{x}{\log x}\cdot\frac{1}{(k-1)!}\cdot\left(\log_2 x + O(1)\right)^{k-1},
$$

an inequality which together with Stirling's formula implies that

$$\#\{n \le x \ : \ \omega(n) = k\} \ll \frac{x}{\log x} \cdot \left(\frac{e \log_2 x + O(1)}{k-1}\right)^{k-1} < \frac{x}{\log x} \cdot \frac{1}{2^{k-1}}$$

since $k - 1 > 3e \log\log x - 1$ and x is assumed to be large. Thus,

$$\#\mathcal{D}_1 = \#\{n \le x \ : \ \omega(n) > 3e \log_2 x\} \ll \frac{x}{\log x} \sum_k \frac{1}{2^k} \ll \frac{x}{\log x} \ll \frac{x}{\log_2^2 x}.$$

Assume now that $\mathcal{D}_2$ is the set of all $n \le x$ which are divisible by the square of a prime $p > \log_2^2 x$. Then

$$\#\mathcal{D}_2 \le \sum_{p > \log_2^2 x} \frac{x}{p^2} \ll \frac{x}{\log_2^2 x}.$$

Let $\mathcal{D}_3$ be the set of those $n \le x$ which are divisible by a prime number p such that $\omega(p-1) \ge b := \lfloor e^2 \log_2 x \rfloor$. Then

$$\begin{aligned} \#\mathcal{D}_3 \quad &\le \quad \sum_{\substack{p \le x \\ \omega(p-1) \ge b}} \frac{x}{p} \le x \sum_{k \ge b} \frac{1}{k!} \left(\sum_{q^a \le x} \frac{1}{q^a}\right)^k \\ &\ll \quad x \sum_{k \ge b} \left(\frac{e \log_2 x + O(1)}{k}\right)^k \ll x \sum_{k \ge b} \frac{1}{2^k} \ll \frac{x}{2^b} \ll \frac{x}{\log x} \ll \frac{x}{\log_2^2 x}, \end{aligned}$$

where we used the facts that $e > 2$ and $2^{e^2} > e$. Let $\mathcal{D}_4$ be the set of those $n \le x$ which are divisible by a prime number p such that $\omega(p+1) \ge b$. The same argument as above shows that

$$\#\mathcal{D}_4 \ll \frac{x}{\log_2^2 x}.$$

Let $\mathcal{D}_5$ be the set of those $n \le x$ which do not belong to $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ and such that there exists a prime power $p^a | n$, where $a = \lfloor c_3 \log_3 x \rfloor$, where $c_3 = \frac{2}{\log 2}$. By an argument similar to the one used in the proof of Lemma 5, we get that

$$\#\mathcal{D}_5 \le x \sum_{p \ge 2} \frac{1}{p^a} \ll \frac{x}{2^a} + x \int_2^\infty \frac{dt}{t^a} \ll \frac{x}{2^a} \ll \frac{x}{\log_2^2 x}.$$

Thus, $\mathcal{D} = \cup_{i=1}^5 \mathcal{D}_i$ contains $O(x/\log_2^2 x)$ elements, as claimed.

§4. The maximal order of $H(n)$

We will show that for n sufficiently large,

$$H(n) \le (1+o(1))\, e^{2\gamma} \log_2^2 n. \tag{6}$$

Then clearly the proof of Theorem 1 will follow if we can also show the following result.

> Claim. *There exists an infinite sequence of integers n for which $H(n)$ is bounded below by $(1+o(1))\, e^{2\gamma} \log_2^2 n$.*

To prove (6), first observe that it follows from Lemma 2 that

$$\sigma(\phi(n)) \le (1+o(1))e^{\gamma}\phi(n)\log_2\phi(n) \le (1+o(1))e^{\gamma}\, n \log_2 n. \tag{7}$$

On the other hand, it follows from Lemma 1 that

$$\phi(n) \ge (1+o(1))\frac{e^{-\gamma}n}{\log_2 n},$$

so that

$$\phi(\sigma(n)) \ge (1+o(1))\frac{e^{-\gamma}\sigma(n)}{\log_2\sigma(n)} \ge (1+o(1))e^{-\gamma}\frac{n}{\log_2 n}. \tag{8}$$

Combining (7) and (8), we obtain (6).

Hence, in order to complete the proof of Theorem 1, it remains to prove our Claim. So let x be a large integer, and let P and Q be the smallest primes such that

$$P \equiv 1 \pmod{M(x)} \qquad \text{and} \qquad Q \equiv -1 \pmod{M(x)},$$

where $M(x) = \mathrm{LCM}[1,2,\dots,x]$, and set

$$n = PQ.$$

From the Prime Number Theorem, it is clear that

$$M(x) = e^{(1+o(1))x} < e^{2x},$$

say. Hence, from Theorem A, it follows that

$$P \ll e^{11x}, \quad Q \ll e^{11x}, \text{ so that } n = PQ \ll e^{22x}.$$

Thus, $n < e^{23x}$ holds for large x. For this particular integer n, we have, since $\phi(n) = (P-1)(Q-1)$,

$$\begin{aligned} \frac{\sigma(\phi(n))}{\phi(n)} &= \prod_{p^{\alpha_p} \| (P-1)(Q-1)} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{\alpha_p}}\right) \\ &\geq \prod_{p^{\alpha_p} \| (P-1)} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{\alpha_p}}\right) \\ &\geq \prod_{p^{\beta_p} \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{\beta_p}}\right), \end{aligned}$$

where each exponent β_p is the unique positive integer satisfying $p^{\beta_p} \leq x < p^{\beta_p+1}$. Therefore,

$$\frac{\sigma(\phi(n))}{\phi(n)} \geq \prod_{p \leq x} \left(1 + \frac{1}{p-1}\right) \prod_{p^{\beta_p} \leq x} \left(1 + O\left(\frac{1}{p^{\beta_p+1}}\right)\right). \tag{9}$$

However,

$$\begin{aligned} \prod_{p^{\beta_p} \leq x} \left(1 + O\left(\frac{1}{p^{\beta_p+1}}\right)\right) &= \exp\left\{O\left(\sum_{p^{\beta_p} \leq x} \frac{1}{p^{\beta_p+1}}\right)\right\} = \exp\left\{O\left(\frac{\pi(x)}{x}\right)\right\} \\ &= 1 + O\left(\frac{1}{\log x}\right). \end{aligned}$$

Using this in (9), we obtain that, by Lemma 1,

$$\frac{\sigma(\phi(n))}{\phi(n)} \geq (1 + o(1)) \prod_{p \leq x} \frac{p}{p-1} = (1 + o(1)) e^{\gamma} \log x. \tag{10}$$

On the other hand, $\sigma(n) = (P+1)(Q+1)$, so that

$$\begin{aligned} \frac{\phi(\sigma(n))}{\sigma(n)} &= \prod_{p \mid (P+1)(Q+1)} \left(1 - \frac{1}{p}\right) \leq \prod_{p \mid (P+1)} \left(1 - \frac{1}{p}\right) \\ &\leq \prod_{p \mid M(x)} \left(1 - \frac{1}{p}\right) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = (1 + o(1)) \frac{e^{-\gamma}}{\log x}, \end{aligned} \tag{11}$$

where we used Lemma 1.

Gathering (10) and (11), we get that

$$H(n) \cdot \frac{\sigma(n)}{\phi(n)} \geq (1+o(1))e^{2\gamma} \log^2 x. \tag{12}$$

Since by our choice of n, we have $\exp\{(1+o(1))x\} < n < \exp\{23x\}$, it follows that $(1+o(1))x < \log n < 23x$ and therefore that $\log_2 n = \log x + O(1)$, which means that (12) can be replaced by

$$H(n) \cdot \frac{\sigma(n)}{\phi(n)} \geq (1+o(1))e^{2\gamma} \log_2^2 n. \tag{13}$$

Observing now that, for large x (that is, large P and Q),

$$\frac{\sigma(n)}{\phi(n)} = \frac{(P+1)(Q+1)}{(P-1)(Q-1)} = 1 + o(1),$$

we conclude that our Claim follows immediately from (13), since then by varying x one obtains infinitely many such integers n. The proof of Theorem 1 is thus complete.

§5. The minimal order of $H(n)$

It follows from Theorem B and Lemma 3 that, for n sufficiently large,

$$\frac{\sigma(\phi(n))}{n} > \kappa \qquad \text{and} \qquad \frac{n}{\sigma(n)} \geq \frac{(1+o(1))e^{-\gamma}}{\log_2 n}.$$

Combining these with the trivial inequality $\dfrac{\sigma(n)}{\phi(\sigma(n))} \geq 1$, we immediately get that

$$H(n) \log_2 n = \frac{\sigma(\phi(n))}{n} \cdot \frac{\sigma(n)}{\phi(\sigma(n))} \cdot \frac{n}{\sigma(n)} \cdot \log_2 n \geq e^{-\gamma}\kappa. \tag{14}$$

To complete the proof of Theorem 2, we shall use an argument developed by Mąkowski and Schinzel in [9].

Let x be large and let $N(x) = \prod_{p<x} p$. Moreover let q be the smallest prime number exceeding $x \log x$, and choose $n = N(x)^{q-1}$, so that

$$\phi(n) = N(x)^{q-2}\phi(N(x)) = \prod_{p<x} p^{\alpha_p},$$

where $\alpha_p = q - 2 + \gamma_p$ and $\gamma_p \geq 0$ is such that $p^{\gamma_p} || \phi(N(x))$. We then have

$$\sigma(\phi(n)) = \prod_{p<x} \sigma(p^{\alpha_p}) = \prod_{p<x} \frac{p^{\alpha_p+1} - 1}{p - 1}$$

and

$$\frac{\sigma(\phi(n))}{\phi(n)} = \prod_{p<x} \frac{p^{\alpha_p+1} - 1}{p^{\alpha_p}(p-1)} = (1 + o(1))e^{\gamma} \log x, \tag{15}$$

by Lemma 1.

On the other hand, again by Lemma 1,

$$\frac{\phi(n)}{n} = \prod_{p<x} \left(1 - \frac{1}{p}\right) = (1 + o(1))\frac{e^{-\gamma}}{\log x}. \tag{16}$$

Combining (15) and (16), we obtain that

$$\frac{\sigma(\phi(n))}{n} = \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n} = 1 + o(1)). \tag{17}$$

We now examine the expression

$$\sigma(n) = \prod_{p<x} \frac{p^q - 1}{p - 1}. \tag{18}$$

Fix a prime $p < x$ and set

$$\frac{p^q - 1}{p - 1} = r_1^{\beta_1} r_2^{\beta_2} \dots r_t^{\beta_t},$$

where, for each $i = 1, 2, \dots, t$, $r_i = r_i(p)$ is a prime and $\beta_i = \beta_i(p)$ a positive integer. We then have $p^q \equiv 1 \pmod{r_i}$ for each positive integer $i \leq t$, and by Fermat's Little Theorem it follows easily that $r_i \equiv 1 \pmod{q}$ (for if not, then from $p^q \equiv 1 \pmod{r_i}$ it would follow that $p \equiv 1 \pmod{r_i}$, which would lead to the conclusion that $(p^q - 1)/(p - 1)$ and $p - 1$ have a common factor $r_i > 1$, which is impossible because $(p^q - 1)/(p-1)$ is congruent modulo $p-1$ to the prime $q > p-1$). Hence

$$x^q > p^q > \frac{p^q - 1}{p - 1} > q^t,$$

which, since $q > x \log x$, implies that

$$t < \frac{q \log x}{\log q} < q. \tag{19}$$

From this it follows that

$$\frac{\phi(\frac{p^q-1}{p-1})}{\frac{p^q-1}{p-1}} = \prod_{i=1}^{t} \left(1 - \frac{1}{r_i}\right) \geq \exp\left\{-2\sum_{i=1}^{t} \frac{1}{r_i}\right\}, \tag{20}$$

where we used the fact that the inequality $1 - z > e^{-2z}$ holds for all z in the interval $(0, 1/4)$. Since it follows from (19) that there are at most q such primes r_i in the arithmetic progression 1 mod q, we have that

$$\sum_{i=1}^{t} \frac{1}{r_i} \leq \frac{1}{q \cdot 1} + \frac{1}{q \cdot 2} + \ldots + \frac{1}{q \cdot q} < \frac{2 \log q}{q},$$

which, combined with (20), yields

$$\frac{\phi(\frac{p^q-1}{p-1})}{\frac{p^q-1}{p-1}} \geq \exp\left\{-\frac{4 \log q}{q}\right\}.$$

It follows from this that

$$1 \geq \frac{\phi(\sigma(n))}{\sigma(n)} \geq \prod_{\substack{r \mid \frac{p^q-1}{p-1} \\ \text{for some } p<x}} \left(1 - \frac{1}{r}\right) \geq \exp\left\{-4\frac{\pi(x)\log q}{q}\right\} = 1 + o(1),$$

since we have chosen $q > x \log x$ and since $\pi(x) \ll x/\log x$. We have thus established that

$$\frac{\phi(\sigma(n))}{\sigma(n)} = 1 + o(1).$$

It now follows by Lemma 1 that

$$
\begin{aligned}
(21)\quad \frac{\phi(\sigma(n))}{n} &= \frac{\phi(\sigma(n))}{\sigma(n)} \cdot \frac{\sigma(n)}{n} = (1+o(1)) \prod_{p<x} \left(1 + \frac{1}{p} + \ldots + \frac{1}{p^{q-1}}\right) \\
&= (1+o(1)) \prod_{p<x} \left\{ \left(1 + \frac{1}{p-1}\right) \left(1 + O\left(\frac{1}{p^q}\right)\right) \right\} \\
&= (1+o(1)) e^{\gamma} (\log x) \exp\left(O\left(\sum_{p<x} \frac{1}{p^q} \right) \right) \\
&= (1+o(1)) e^{\gamma} (\log x) \exp\left(O\left(\frac{\pi(x)}{2^{x \log x}} \right) \right) = (1+o(1)) e^{\gamma} \log x.
\end{aligned}
$$

Combining (17) and (21), we get

$$
(22) \qquad H(n) = \frac{\sigma(\phi(n))}{n} \cdot \frac{n}{\phi(\sigma(n))} = (1+o(1)) \frac{e^{-\gamma}}{\log x}.
$$

It remains to estimate the size of n. Recall that, by our choice of n and q, we have

$$
n = \left(\prod_{p<x} p \right)^{q-1} = \exp\{(1+o(1)) xq\} = \exp\{(1+o(1)) x^2 \log x\},
$$

so that $(1+o(1))x^2 \log x = \log n$, from which we easily obtain that

$$
x = (1+o(1)) \sqrt{\frac{2 \log n}{\log_2 n}},
$$

which yields

$$
\log x = \frac{1}{2}(1+o(1)) \log_2 n.
$$

Substituting this in (22), we obtain that

$$
H(n) = (1+o(1)) \frac{e^{-\gamma}}{\frac{1}{2} \log_2 n},
$$

from which we may conclude that there exist infinitely many integers n such that

$$
H(n) \log_2 n = (1+o(1)) 2e^{-\gamma}.
$$

Combining this last result with (14) and taking into account the remark following the statement of Theorem B concerning the improved lower bound for κ, the proof of Theorem 2 is complete.

§6. The mean value of $H(n)$

We use the method developed in [8]. Let $M_0(x)$ be the least common multiple of all prime powers $p^a < g(x)$, where $g(x) = c_1 \log_2 x / \log_3 x$ and c_1 is the constant mentioned in Lemma 4. Moreover, let $\mathcal{A} = \mathcal{A}(x) = \{n : \sqrt{x} < n \le x \text{ and } M_0(n) | \gcd(\phi(n), \sigma(n))\}$. Then

$$\frac{\sigma(\phi(n))}{\phi(n)} \ge e^\gamma \log_3 x \left(1 + O\left(\frac{1}{\log_3 x}\right)\right) \qquad (n \in \mathcal{A}). \tag{23}$$

(This follows from inequality (37) in [8].) Using the same method and then applying Lemma 1, we get

$$\frac{\phi(\sigma(n))}{\sigma(n)} \le \frac{\phi(M_0(n))}{M_0(n)} = \prod_{p<g(x)} \left(1 - \frac{1}{p}\right) \le \frac{e^{-\gamma}}{\log_3 x}\left(1 + O\left(\frac{1}{\log_3 x}\right)\right) \qquad (n \in \mathcal{A}). \tag{24}$$

Combining (23) and (24) yields

$$H(n) \ge \frac{\phi(n)}{\sigma(n)} \frac{e^\gamma \log_3 x}{e^{-\gamma}/\log_3 x}\left(1 + O\left(\frac{1}{\log_3 x}\right)\right) = \frac{\phi(n)}{\sigma(n)} e^{2\gamma} \log_3^2 x \left(1 + O\left(\frac{1}{\log_3 x}\right)\right) \tag{25}$$

for $n \in \mathcal{A}$.

It follows from this that

$$\sum_{n \le x} H(n) \ge \sum_{\substack{n \le x \\ n \in \mathcal{A}}} H(n) \ge e^{2\gamma} \log_3^2 x \left(1 + O\left(\frac{1}{\log_3 x}\right)\right) \sum_{\substack{n \le x \\ n \in \mathcal{A}}} \frac{\phi(n)}{\sigma(n)}. \tag{26}$$

Now using Lemma 4 to estimate the size of $[1, x] \setminus \mathcal{A}$ and using the fact that $\phi(n) \le \sigma(n)$ holds for all n, we get by Lemma 6

$$\begin{aligned} \sum_{\substack{n \le x \\ n \in \mathcal{A}}} \frac{\phi(n)}{\sigma(n)} &\ge \sum_{n \le x} \frac{\phi(n)}{\sigma(n)} - (x - \#\mathcal{A}) = \sum_{n \le x} \frac{\phi(n)}{\sigma(n)} + O\left(\frac{x}{\log_3^2 x}\right) \\ &= c_0 x + O\left(\frac{x}{\log_3^2 x}\right). \end{aligned}$$

Combining this with (26), it follows that

$$\sum_{n\le x} H(n) \ge c_0\, e^{2\gamma} x\, \log_3^2 x + O\left(x \log_3 x\right). \tag{27}$$

It remains to obtain the corresponding upper bound for $\sum_{n\le x} H(n)$. To do so, we first observe that we only need to consider those integers $n\in[\sqrt{x}, x]$, since it follows from Theorem 1 that

$$\sum_{n\le \sqrt{x}} H(n) = O\left(\sqrt{x}\, \log_2^2 x\right). \tag{28}$$

Consider now the set

$$\mathcal{B} = \mathcal{B}(x) = \left\{ n : \sqrt{x} < n \le x,\ h_\phi(n) < \frac{1}{\sqrt{\log_3 x}},\ h_\sigma(n) < \frac{1}{\sqrt{\log_3 x}} \right\},$$

and given a positive integer $n\in\mathcal{B}$, write $\phi(n) = n_1\cdot n_2$, where

$$n_1 = \prod_{\substack{p^{\alpha_p}\,\|\,\phi(n)\\ p\le \log_2 x}} p^{\alpha_p} \text{ and } n_2 = \prod_{\substack{p^{\alpha_p}\,\|\,\phi(n)\\ p> \log_2 x}} p^{\alpha_p},$$

so that, using Lemma 1,

$$\begin{aligned}
\frac{\sigma(\phi(n))}{\phi(n)} &= \prod_{p|n_1}\left(1+\frac{1}{p}+\ldots+\frac{1}{p^{\alpha_p}}\right)\cdot \prod_{p|n_2}\left(1+\frac{1}{p}+\ldots+\frac{1}{p^{\alpha_p}}\right)\\
&\le \left(e^\gamma \log_3 x + O(1)\right)\cdot \exp(O(h_\phi(n)))\\
&= \left(e^\gamma \log_3 x + O(1)\right)\cdot \exp\left\{O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right\}\\
&= e^\gamma \log_3 x + O\left(\sqrt{\log_3 x}\right) \qquad (n\in\mathcal{B}).
\end{aligned} \tag{29}$$

On the other hand, given $n\in\mathcal{B}$ and writing $\sigma(n) = m_1\cdot m_2$, where

$$m_1 = \prod_{\substack{p^{\alpha_p}\,\|\,\sigma(n)\\ p\le \log_2 x}} p^{\alpha_p} \quad \text{and} \quad m_2 = \prod_{\substack{p^{\alpha_p}\,\|\,\sigma(n)\\ p> \log_2 x}} p^{\alpha_p},$$

we get, by a similar argument,

$$\begin{aligned}(30)\qquad \frac{\phi(\sigma(n))}{\sigma(n)} &= \frac{\phi(m_1)}{m_1}\cdot\frac{\phi(m_2)}{m_2}\\ &\geq \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{1}{\log_3 x}\right)\right)\cdot\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\\ &= \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right) \qquad (n\in\mathcal{B}).\end{aligned}$$

Gathering (29) and (30), we obtain

$$(31)\qquad H(n)\leq \frac{\phi(n)}{\sigma(n)}\, e^{2\gamma}\,\log_3^2 x\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\qquad (n\in\mathcal{B}),$$

from which it follows that

$$\begin{aligned}(32)\qquad \sum_{\substack{n\leq x\\ n\in B}} H(n) &\leq e^{2\gamma}\,\log_3^2 x\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\sum_{\substack{n\leq x\\ n\in B}}\frac{\phi(n)}{\sigma(n)}\\ &\leq e^{2\gamma}c_0 x\,\log_3^2 x+O(x\log_3^{3/2} x).\end{aligned}$$

It remains to consider the contribution of those integers $n\in[\sqrt{x},x]$ which do not belong to the set $\mathcal{B}$. The set of these numbers is contained in $\mathcal{C}_\phi\cup\mathcal{C}_\sigma$, where, given $f\in\{\phi,\sigma\}$, we write $\mathcal{C}_f$ for the set of those numbers $n\in[\sqrt{x},x]$ such that $h_f(n)\geq 1/\sqrt{\log_3 x}$. Lemma 5 shows that

$$\frac{x}{\log_3 x}\gg\sum_{n\in C_f} h_f(n)\geq\frac{\#\mathcal{C}_f}{\sqrt{\log_3 x}},$$

so that

$$(33)\qquad \#\mathcal{C}_f\ll x/\sqrt{\log_3 x}\qquad \text{for } f=\phi \text{ and } f=\sigma.$$

We now call upon Lemma 7. Let $\mathcal{D}$ be the set mentioned in the proof of that lemma. Since by Theorem 1, $H(n)\ll\log_2^2 n$, it follows that

$$(34)\qquad \sum_{n\in\mathcal{D}} H(n)=O(x).$$

We now let $\mathcal{E}$ be the set of those $n \le x$ which are not in $\mathcal{D}$. It is easy to see that if $n \in \mathcal{E}$, then both inequalities $\omega(\phi(n)) < c_4 \log_2^2 x$ and $\omega(\sigma(n)) < c_4 \log_2^2 x$ hold for large x, where c_4 can be chosen to be any constant $> 4e^3$. Indeed, since $\omega(n) < 3e \log_2 x$, it follows that if $\omega(\phi(n)) > 4e^3 \log_2 x$, then there must exist a prime $p|n$ such that $\omega(p-1) > e^2 \log_2 x$, for if not, that is, if $\omega(p-1) \le e^2 \log_2 x$ holds for all prime factors p of n, then

$$\omega(\phi(n)) \le \omega(n)(1 + \sum_{p|n} \omega(p-1)) \le 3e(1 + e^2 \log_2 x) \log_2 x < 4e^3 \log_2^2 x.$$

However, the inequality $\omega(p-1) > e^2 \log_2 x$ cannot hold for some prime divisor p of n because n is not in $\mathcal{D}_3$. Write $n = n' \cdot n''$, where

$$n' = \prod_{\substack{p^{\alpha_p} || n \\ \alpha_p > 1}} p^{\alpha_p} \quad \text{and} \quad n'' = \prod_{p||n} p.$$

Since $n \notin \mathcal{D}$, the same argument as above shows that $\omega(\sigma(n'')) \ll \log_2^2 x$. Finally, note that

$$n' < \exp(O((\log_3 x)\pi(\log_2^2 x)) = \exp(O(\log_2^2 x)),$$

so that

$$\sigma(n') < \exp(O(\log_2^2 x)),$$

which shows that $\omega(\sigma(n')) = o(\log_2^2 x)$. Thus, if $n \in \mathcal{E}$, then $\omega(\phi(n))$ and $\omega(\sigma(n))$ are both $O(\log_2^2 x)$. In particular, for large x, we have that

$$\max\{h_\phi(n), h_\sigma(n)\} \le \sum_{\log_2 x < p < \log_2^3 x} \frac{1}{p} \ll 1.$$

Hence, writing $\phi(n) = n_1 \cdot n_2$ and $\sigma(m) = m_1 \cdot m_2$ as previously, we get that for $n \in \mathcal{E}$,

$$\begin{aligned} \frac{\sigma(\phi(n))}{\phi(n)} &= \prod_{p^{\alpha_p}||n_1} \left(1 + \frac{1}{p} + \ldots + \frac{1}{p^{\alpha_p}}\right) \prod_{p^{\alpha_p}||n_2} \left(1 + \frac{1}{p} + \ldots + \frac{1}{p^{\alpha_p}}\right) \\ &\le \prod_{p \le \log_2 x} \left(1 + \frac{1}{p-1}\right) \exp(O(h_\phi(n)) \ll \log_3 x \end{aligned}$$

and

$$\begin{aligned}\frac{\phi(\sigma(n))}{\sigma(n)} &= \prod_{p||m_1}\left(1-\frac{1}{p}\right)\prod_{p||m_2}\left(1-\frac{1}{p}\right)\\ &\geq \prod_{p\leq \log_2 x}\left(1-\frac{1}{p}\right)\exp(-h_\sigma(n)) \gg \log_3 x,\end{aligned}$$

from which we may conclude that $H(n) \ll \log_3^2 x$ holds for all $n \in \mathcal{E}$. Finally, recall that by (33), the set of those $n \in \mathcal{C}_\phi \cup \mathcal{C}_\sigma$ is of cardinality at most $O(x/\sqrt{\log_3 x})$, and therefore that

$$\sum_{n\in(\mathcal{C}_\phi\cup\mathcal{C}_\sigma)\cap\mathcal{E}} H(n) \leq \max_{n\in\mathcal{E}}\{H(n)\}\cdot\#(\mathcal{C}_\phi\cup\mathcal{C}_\sigma) \ll x\log_3^{3/2} x,$$

which together with (28), (32) and (34) shows that

$$\sum_{n\leq x} H(n) \leq e^{2\gamma}c_0 x\log_3^2 x + O(x\log_3^{3/2} x). \tag{35}$$

Combining (27) and (35) completes the proof of Theorem 3.

§7. The first density theorem for $H(n)$

Here, we follow essentially an argument used in [8]. In view of (25) and (31), it follows that both inequalities

$$\begin{aligned}H(n) &\geq (1+o(1))\frac{\phi(n)}{\sigma(n)}e^{2\gamma}\log_3^2 n\\ H(n) &\leq (1+o(1))\frac{\phi(n)}{\sigma(n)}\ e^{2\gamma}\log_3^2 n\end{aligned}$$

hold on a set of density 1. Therefore, it follows that on a set of density 1,

$$H(n) = (1+o(1))e^{2\gamma}\log_3^2 n\frac{\phi(n)}{\sigma(n)}.$$

Since $\phi(n)/\sigma(n)$ has a continuous distribution function (see Exercices 2 and 3 of Chapter III.2 in Tenenbaum [13]), the proof of Theorem 4 is complete.

§8. The second density theorem for $H(n)$

Fix $\delta \in (0, \infty)$ and let x be a very large positive real number. We shall now construct a finite set of primes $\mathcal{R}$ such that each of its element is larger than than x^{x^2} and with the property that

$$\prod_{r \in \mathcal{R}} \left(1 + \frac{1}{r}\right) \in \left(\frac{e^\gamma \delta \log x}{2} - 1, \frac{e^\gamma \delta \log x}{2} + 1\right).$$

To construct this set $\mathcal{R}$, let $r_1 < r_2 < \ldots$ be all the primes $> x^{x^2}$ and let k be the largest positive integer such that

$$\prod_{i=1}^{k} \left(1 + \frac{1}{r_i}\right) \le \frac{e^\gamma \delta \log x}{2}.$$

Observe that by the maximality of k and the fact that

$$r_{k+1} \ge r_1 > x^{x^2} > \frac{e^\gamma \delta \log x}{2}$$

holds for all x sufficiently large, we get that

$$\prod_{i=1}^{k+1} \left(1 + \frac{1}{r_i}\right) \in \left(\frac{e^\gamma \delta \log x}{2}, \frac{e^\gamma \delta \log x}{2} + 1\right).$$

Hence, we can take $\mathcal{R} = \{r_i \ : \ i = 1, \ldots, k+1\}$. Note that since

$$\prod_{i=1}^{k+1} \left(1 + \frac{1}{r_i}\right) = \exp(\log_2 r_{k+1} - \log_2 r_1 + o(1)) > \exp(\log_2 r_{k+1} - 3 \log x),$$

it follows that the inequality $r_{k+1} < e^{x^4}$ holds for large x, for if not, then $r_k \ge e^{x^4}/2$, in which case

$$\prod_{i=1}^{k} \left(1 + \frac{1}{r_i}\right) > \exp(\log_2 r_k - \log_2 r_1 + o(1)) > \exp(\log x) = x > \frac{e^\gamma \delta \log x}{2} + 1$$

which contradicts the definition of k.

We now let y be a parameter that depends on x and such that $z := \log_2 y > r_{k+1}$. This inequality is fulfilled if we choose $\log_2 y > e^{x^4}$, which in turn holds if $\log_3 y > x^4$. Then let $\mathcal{P}$ be the set of all primes $p \le y$ such that $p \equiv 13 \pmod{72}$, $p \equiv 1 + r_i \pmod{r_i^2}$ for all $i = 1, \ldots, k+1$, and both $p-1$ and $p+1$ are coprime to all primes $r \le z$ which are ≥ 5 and which do not belong to $\mathcal{R}$. Observe that the above conditions certainly put p in an arithmetic progression $a \pmod{b}$, where

$$b = 72 \prod_{i=1}^{k+1} r_i^2,$$

and $a \equiv 13 \pmod{72}$ and $a \equiv 1 + r_i \pmod{r_i^2}$ for $i = 1, \ldots, k+1$.

Now let

$$T := \prod_{\substack{5 \le r \le z \\ r \notin \mathcal{R}}} r,$$

and, for each $d|T$, let

$$\mathcal{A}(d) := \{a_d \pmod{bd} : d|a_d^2 - 1 \text{ and } a_d \equiv a \pmod{b}\},$$

so that $\#\mathcal{A}(d) = 2^{\omega(d)}$.

By the principle of inclusion and exclusion, the cardinality of the set of primes $\mathcal{P}$ is none other than

$$\sum_{d|T} \mu(d) \sum_{a_d \in \mathcal{A}(d)} \pi(y; a_d, bd),$$

where, as usual, $\pi(y; s, t)$ stands for the number of primes $p \le y$ satisfying $p \equiv s \pmod{t}$. Observing that $bT \ll \prod_{r \le z} r^2 \le e^{2(1+o(1))z} < e^{3z} < y^{1/3}$, we get, by the Bombieri-Vinogradov Theorem, that

$$\mathcal{P} = \frac{\pi(y)}{\phi(b)} \prod_{\substack{5 \le r \le z \\ r \notin \mathcal{R}}} \left(1 - \frac{2}{r-1}\right) + O\left(\frac{2^{\pi(z)} y}{\log^{10} y}\right).$$

Since

$$2^{\pi(z)} = \exp(O(\log_2 y / \log_3 y)) = (\log y)^{o(1)},$$

while

$$\phi(b) \ll \prod_{r \le z} r^2 < \exp(3 \log_2 y) = \log^3 y,$$

and since

$$\prod_{r\le z}\left(1-\frac{2}{r-1}\right) \gg \exp\left\{-\sum_{r\le z}\frac{2}{r-1}\right\} \gg \exp\{-2\log\log z\} = \frac{1}{\log^2 z} \ge \frac{1}{\log y},$$

it follows that

$$\mathcal{P} = \frac{\pi(y)}{\phi(b)} \prod_{\substack{5\le r\le z\\ r\notin\mathcal{R}}} \left(1-\frac{2}{r-1}\right) + O\left(\frac{y}{\log^9 y}\right) \gg \frac{\pi(y)}{\log^4 y} \gg \frac{y}{\log^5 y}.$$

Finally, let $\mathcal{P}'$ be the subset of those primes $p \in \mathcal{P}$ such that neither of $\omega(p-1)$ or $\omega(p+1)$ is larger than $e^2 \log_2 y$. From the estimates due to Hardy and Ramanujan (see [5] and the proof of Theorem 3), we know that

$$\begin{aligned}
\#\{n \le y \ :\ \omega(n) > e^2 \log_2 y\} &\ll \frac{y}{\log y} \sum_{k > e^2 \log_2 y} \frac{1}{(k-1)!} (\log_2 y + O(1))^k \\
&\ll \frac{y}{\log y} \sum_{k > e^2 \log_2 y} \left(\frac{e \log_2 y + O(1)}{k}\right)^k \\
&\ll \frac{y}{\log y} \cdot \frac{1}{2^{e^2 \log_2 y}} = o\left(\frac{y}{\log^5 y}\right),
\end{aligned}$$

because $e^2 \log 2 + 1 > 5$. Thus,

$$\#\mathcal{P}' \gg \frac{y}{\log^5 y}.$$

In particular, $\mathcal{P}'$ is non empty for large y. Select P in $\mathcal{P}'$ and let $n = N(x)P$, where $N(x) = \prod_{p<x} p$. Then, $\phi(n) = 12\phi(N(x)) \cdot (P-1)/12$, $\sigma(n) = 2\sigma(N(x)) \cdot (P+1)/2$, and $(P-1)/12$ is coprime to $4\phi(N(x))$, while $(P+1)/2$ is coprime to $2\sigma(N(x))$. The arguments from the proof of Theorem 2 now immediately show that

$$\begin{aligned}
\frac{\sigma(\phi(n))}{\phi(n)} &= \frac{\sigma(12N(x))}{12N(x)} \cdot \frac{\sigma((P-1)/12)}{(P-1)/12} \\
&= e^\gamma \log x \left(1 + O\left(\frac{1}{\log x}\right)\right) \prod_{r\in\mathcal{R}} \left(1+\frac{1}{r}\right) \prod_{\substack{r^{\alpha_r} \| P-1\\ r>z}} \left(1+\frac{1}{r}+\ldots+\frac{1}{r^{\alpha_r}}\right) \\
&= e^\gamma \log_2 x \left(1 + O\left(\frac{1}{\log x}\right)\right) \cdot \frac{e^\gamma \delta \log x}{2} \cdot \exp\left(O\left(\sum_{\substack{r|(P-1)\\ r>z}} \frac{1}{r}\right)\right).
\end{aligned}$$

Noting that $P-1$ has no more that $e^2 \log_2 y$ prime factors, it follows easily that

$$\begin{aligned}\sum_{\substack{r|(P-1)\\ r>z}} \frac{1}{r} &\le \sum_{\log_2 y<r<\log_2 y \log_3^2 y} \frac{1}{r} \ll \log\left(\frac{\log_3 y + 2\log_4 y}{\log_3 y}\right)\\ &\ll \frac{\log_4 y}{\log_3 y} \ll \frac{\log x}{x^4} \ll \frac{1}{\log x}.\end{aligned}$$

This means that

$$\frac{\sigma(\phi(n))}{\phi(n)} = \frac{e^\gamma \delta}{2} \log^2 x \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

By similar arguments, we get that

$$\frac{\phi(n)}{n} = \prod_{r\le x}\left(1-\frac{1}{r}\right)\cdot \frac{P-1}{P} = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

It follows that

$$\frac{\sigma(\phi(n))}{n} = \frac{e^\gamma \delta}{2} \log x \left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{36}$$

As we obtained (17) in the proof of Theorem 2, we also get, handling the case $P+1$ as we did in the case $P-1$,

$$\begin{aligned}\frac{\phi(\sigma(n))}{\sigma(n)} &= \frac{1}{2}\frac{\phi(\sigma(N(x)))}{\sigma(N(x))}\cdot\frac{\phi((P+1)/2)}{(P+1)/2}\\ &= \frac{1}{2}\left(1 + O\left(\frac{1}{\log x}\right)\right)\cdot\prod_{\substack{r|(P+1)\\ r>z}}\left(1-\frac{1}{r}\right)\\ &= \frac{1}{2}\left(1 + O\left(\frac{1}{\log x}\right)\right).\end{aligned} \tag{37}$$

Finally,

$$\frac{\sigma(n)}{n} = \frac{\sigma(N(x))}{N(x)}\cdot\frac{P+1}{P} = e^\gamma \log x\left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{38}$$

Gathering (37) and (38) yields

$$\frac{\phi(\sigma(n))}{n} = \frac{e^{\gamma}\log x}{2}\left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{39}$$

Combining (36) and (39), we obtain

$$H(n) = \frac{e^{\gamma}\delta\log x}{2}\cdot\frac{2}{e^{\gamma}\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right) = \delta\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Since x is arbitrary, we get that δ is a cluster point of $\{H(n)\}_{n\geq 1}$, as claimed.

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JMDK, le 29 décembre 2006; fichier: phisigmajmdk9.tex