Title: On an estimate of Kanold

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Abstract: Given a positive integer n, let $d(n)$ (resp. $\sigma(n)$) stand for the number (resp. the sum) of its positive divisors. Letting $E(x)$ stand for the number of positive integers $n \le x$ such that $\gcd(nd(n), \sigma(n)) = 1$, we show that there exists a positive constant c such that $E(x) = c(1 + o(1))\sqrt{\dfrac{x}{\log x}}$, thus improving upon a result of Kanold.

Key words: number of divisors, sum of divisors

AMS Mathematics Subject Classification: 11N37

§1. Introduction

Given a positive integer n, let $d(n)$ (resp. $\sigma(n)$) stand for the number (resp. the sum) of its positive divisors. Letting $E(x)$ stand for the number of positive integers $n \le x$ such that $\gcd(nd(n), \sigma(n)) = 1$, then Kanold [2] has shown that there exist positive constants $c_1 < c_2$ and a positive number x_0 such that

$$c_1 < E(x)/\sqrt{x/\log x} < c_2 \qquad (x \ge x_0). \tag{1}$$

Here we improve (1) by providing an asymptotic estimate for $E(x)$. In fact, we prove the following result.

Theorem. *There exists a positive constant c such that*

$$E(x) = c(1 + o(1))\sqrt{\frac{x}{\log x}}.$$

Acknowlegements: The first author would like to acknowledge the financial support received from NSERC. The second author would like to acknowledge the financial support received from the Applied Number Theory Research Group of the Hungarian Academy of Science and from OTKA.

§2. Preliminary results and notations

Throughout this paper, p, q and π always stand for prime numbers.

We first define the following sets:

$$\begin{aligned} E &= \{n : \gcd(nd(n), \sigma(n)) = 1\} \\ \mathcal{M} &= \{m : m \text{ squarefree}, \ p|m \Rightarrow p \equiv -1 \pmod 3\}. \end{aligned}$$

Lemma 1.

(i) If p is odd and $p\|n$, then $n \notin E$.

(ii) If $p^2\|n$ for some $n \in E$, then $p \equiv -1 \pmod 3$.

(iii) If $n = n_1 n_2 \in E$ with $(n_1, n_2) = 1$, then $n_1 \in E$ and $n_2 \in E$.

Proof. Part (i) follows from the fact that, writing $n = pn_1$ with $(p, n_1) = 1$, then both $d(p)$ and $\sigma(p)$ are even.

To prove part (ii), observe that if $p \equiv 1 \pmod 3$, then $\sigma(p^2) = 1 + p + p^2 \equiv 0 \pmod 3$, in which case, since $d(p^2) = 3$, $n \notin E$.

Part (iii) is obvious.

Lemma 2. *If $m \in \mathcal{M}$, then $m^2 \in E$.*

Proof. Since $m \in \mathcal{M}$, it can be written as $m = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots p_r$ are primes $\equiv -1 \pmod 3$. Therefore

$$d(m^2) = 3^r, \quad \text{while } \sigma(m^2) = \prod_{i=1}^{r}(1 + p_j + p_j^2) \equiv 1 \pmod 3. \tag{2}$$

Fix a positive integer $j \le r$ and let p be a prime divisor of $\sigma(p_j^2)$. We shall prove that $p \equiv 1 \pmod 3$. Since

$$p_j^2 + p_j + 1 \equiv 0 \pmod p, \tag{3}$$

it is clear that $p \neq 2$. But then (3) is successively equivalent to

$$\begin{aligned} 4p_j^2 + 4p_j + 4 &\equiv 0 \pmod p, \\ (2p_j + 1)^2 + 3 &\equiv 0 \pmod p, \end{aligned}$$

so that in particular $\left(\frac{-3}{p}\right) = 1$, thus implying that

$$1 = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}} = \left(\frac{p}{3}\right),$$

and therefore that $p \equiv 1 \pmod 3$, as claimed.

We have thus established that $(m, \sigma(m^2)) = 1$. Hence, in view of (2), the proof of Lemma 2 is complete.

Each integer $n \in E$ can be written as

$$n = Km^2 \qquad (m, K) = 1, \tag{4}$$

where $m = \prod_{\substack{p^2 \| n \\ p \equiv -1 \pmod 3}} p$. Assume that m is the maximal divisor of n with this property. Then the representation (4) is unique and moreover $m \in \mathcal{M}$, $K \in E$. Then, let E^* be the set of those positive integers K for which there exists at least one positive integer $m \in \mathcal{M}$ such that $n = Km^2 \in E$, with K and m as in (4).

Remark 1. Assume that $K \in E^*$ and that $\pi^\gamma \| K$ for some prime π and some positive integer γ. Then $\gamma \ge 3$, except perhaps if $\pi = 2$ or 3, since $E^* \subseteq E$.

§3. Proof of the Theorem

Taking into account Remark 1, it follows that there exists a positive constant d_1 such that

$$\sum_{\substack{K \le Y_0 \\ K \in E^*}} 1 \le d_1 Y_0^{1/3} \qquad (Y_0 \ge 1), \tag{5}$$

and by a classical sieve argument, we obtain that, for some positive constant d_2,

$$\sum_{\substack{m^2 \le x \\ m \in \mathcal{M}}} 1 \le d_2 \sqrt{\frac{x}{\log x}} \qquad (x \ge 2). \tag{6}$$

For each positive integer T, consider the function

$$F_T(s) := \prod_{\substack{p\not\equiv -1 \pmod 3\\ p\nmid T}} \left(1+\frac{1}{p^s}\right) = \prod_{\substack{p\not\equiv -1 \pmod 3\\ p\mid T}} \left(1+\frac{1}{p^s}\right)^{-1} \cdot F_1(s),$$

where

$$F_1(s) = \sum_{m\in\mathcal{M}} \frac{1}{m^s} = \prod_{p\equiv -1 \pmod 3} \left(1+\frac{1}{p^s}\right).$$

Defining

$$\chi(n) = \begin{cases} 1 & \text{if } n\equiv 1 \pmod 3,\\ -1 & \text{if } n\equiv -1 \pmod 3, \end{cases} \qquad L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

we have that

$$\begin{aligned} \log F_1(s) &= \sum_{p\equiv -1 \pmod 3} \frac{1}{p^s} + \sum_{p\equiv -1 \pmod 3} \sum_{\nu=2}^{\infty} \frac{1}{\nu p^{\nu s}} \\ &= \frac{1}{2}\log\zeta(s) - \frac{1}{2}\log L(s,\chi) + u(s), \end{aligned} \tag{7}$$

where $u(s)$ is bounded in the domain $\sigma > \frac{1}{2}+\varepsilon$, for any given $\varepsilon > 0$. Hence, it follows from (7) that

$$F_1(s) = \sqrt{\frac{\zeta(s)}{L(s,\chi)}}\ \exp\{u(s)\},$$

from which one easily obtains (using for instance Theorem 10.1 from the book of Bateman and Diamond [1]) that there exists some positive constant c_1 such that

$$\sum_{\substack{m\le z^{1/2}\\ m\in\mathcal{M}}} 1 = c_1\sqrt{\frac{z}{\log z}}\left(1+O\left(\frac{1}{\sqrt{\log z}}\right)\right).$$

Now, let Y_0 be an arbitrarily large but fixed number.

Let us set $L_K(x) := \#\{n\le x : n = Km^2 \in E\}$, so that $E(x) = \sum_{K\le x} L_K(x)$. Write

$$E(x) = \sum_{\substack{K\le Y_0\\ K\in E^*}} L_K(x) + \sum_{\substack{Y_0<K\le x\\ K\in E^*}} L_K(x) = S_1 + S_2, \tag{8}$$

say.

As we shall see, S_1 provides the main contribution to $E(x)$, while S_2 is negligible. Indeed, we have

$$S_2 \le \sum_{\substack{Y_0<K<x^{3/4}\\ K\in E^*}} L_K(x) + \sum_{\substack{x^{3/4}\le K\le x\\ K\in E^*}} L_K(x) = S_3 + S_4, \tag{9}$$

say. Using (5) and (6), in the range of the sum S_3, we have $L_K(x) \ll \left(\frac{x}{K}\right)^{1/2} \frac{1}{\sqrt{\log x}}$, so that

$$S_3 \ll \left(\frac{x}{\log x}\right)^{1/2} \delta(Y_0), \tag{10}$$

where $\delta(Y_0) := \sum_{\substack{K>Y_0\\ K\in E^*}} \frac{1}{\sqrt{K}}$. Observe that, since $\sum_{K\in E^*} 1/K$ is convergent, we have that $\delta(Y_0)\to 0$ as $Y_0\to\infty$. Hence, it follows from this observation and from (10) that, for some absolute positive constant d_3,

$$S_3 \le d_3\delta(Y_0)\sqrt{\frac{x}{\log x}}. \tag{11}$$

On the other hand, in the range of the sum S_4, $L_K(x) \ll \left(\frac{x}{K}\right)^{1/2}$, so that, using Remark 1,

$$S_4 \ll \frac{x^{1/2}}{x^{1/8}} = x^{3/8}. \tag{12}$$

Collecting (11) and (12) in (9) and (8), we get that

$$|E(x) - S_1| \le d_3\delta(Y_0)\sqrt{\frac{x}{\log x}} + O(x^{3/8}). \tag{13}$$

In order to estimate S_1, we need to assess the size of $L_K(x)$ for $K\le Y_0$, $K\in E^*$.

For each prime $q\equiv 1 \pmod 3$, let $\ell_1(q)$ and $\ell_2(q)$ be the solutions of $u^2+u+1\equiv 0 \pmod q$, and assume that $0<\ell_1(q)<\ell_2(q)<q$. It is clear that $(\ell_i(q),q)=1$ $(i=1,2)$. Consider the following sets:

$$\begin{aligned}\mathcal{T}_K &= \{p : p|\sigma(K),\ p\equiv -1 \pmod 3\},\\ \mathcal{S}_K &= \{q : q|Kd(K),\ q\equiv 1 \pmod 3\}.\end{aligned}$$

We shall now find necessary and sufficient conditions that will ensure that $n=Km^2$ belongs to E. First of all, since $K\in E^*$, $(Kd(K),\sigma(K))=1$. Assuming that $n\ne K$, that is that $m>1$, then $(nd(n),\sigma(n))=1$ holds if and only if $3\not|\sigma(K)$, $(m,\sigma(K))=1$ and $(\sigma(m^2),Kd(K))=1$, since the relation $(m^2d(m^2),\sigma(m^2))=1$ holds by Lemma 2.

Therefore, $m\in\mathcal{M}$ is a suitable integer (as a component of the integer $n=Km^2$ in the format given by (4)) if and only if

(a) m has no factor from the set $\mathcal{T}_K$,

(b) $(p,m)=1$ if $p\equiv \ell_1(q) \pmod q$ or $p\equiv \ell_2(q) \pmod q$ for some $q\in\mathcal{S}_K$.

Now letting

$$G_K(s) := \sum_{\substack{m\\ Km^2\in E}} \frac{1}{m^s},$$

we can therefore write this function in the form

$$G_K(s) = A_K(s)\cdot B_K(s)\cdot C_K(s),$$

where

$$A_K(s) = \prod_{p\equiv -1 \pmod 3}\left(1+\frac{1}{p^s}\right), \qquad B_K(s) = \prod_{p\in\mathcal{T}_K}\left(1+\frac{1}{p^s}\right)^{-1}, \tag{14}$$

$$C_K(s) = \prod_{\substack{\pi \notin \mathcal{T}_K \\ \exists q \in \mathcal{S}_K, \\ \pi \equiv \ell_1(q) \pmod q \\ \pi \equiv \ell_2(q) \pmod q \\ \pi \equiv -1 \pmod 3}} \left(1 + \frac{1}{\pi^s}\right)^{-1} \tag{15}$$

There are two types of $K \in E^*$:

- TYPE 1: $S_K = \emptyset$, i.e. each prime divisor of $Kd(K)$ is 3 or $\equiv -1 \pmod 3$. Then $C_K(s) = 1$, and the right hand side of $B_K(s)$ is a finite product, from which it follows that, for some positive constant d_4,

$$L_K(x) = d_4 \sqrt{\frac{x/K}{\log x}} \prod_{p \in \mathcal{T}_K} \left(1 + \frac{1}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{16}$$

- TYPE 2: $S_K \neq \emptyset$. Then there are infinitely many primes π in the product on the right hand side of (15), and in fact we even have that there exists a positive number η_K such that

$$\sum_{\substack{\pi \le y \\ \pi \in \text{ product of (15)}}} \frac{1}{\pi} \ge \eta_K \log\log y + O_K(1). \tag{17}$$

  Therefore, by the Brun-Selberg sieve, we obtain that

$$\sum_{\substack{m \le x \\ Km^2 \in E}} 1 \ll x \prod_{\substack{p \equiv 1 \pmod 3 \\ \pi \in \text{ product of (15)} \\ \pi \le x}} \left(1 - \frac{1}{p}\right)^{-1} \ll \frac{x}{(\log x)^{\frac{1}{2} + \eta_K}}, \tag{18}$$

  where the constant implicit in $\ll$ may depend on K. Consequently, for those K of Type 2, we have that $L_K(x)/\sqrt{x/\log x} \to 0$ as $x \to \infty$, and this relation holds uniformly for K in the range $[1, Y_0]$.

Hence, we only need to consider those K of Type 1, which allows us to conclude, from (13) and (16), that

$$E(x) = d_4 \sqrt{\frac{x}{\log x}} \sum_{\substack{K \text{ of Type 1} \\ K \le Y_0}} \frac{1}{K^{1/2}} \prod_{p \in \mathcal{T}_K} \left(1 + \frac{1}{p}\right)^{-1} + O\left(\delta(Y_0) \sqrt{\frac{x}{\log x}}\right) + o\left(\sqrt{\frac{x}{\log x}}\right). \tag{19}$$

Now let $D(Y_0) := \sum_{\substack{K \text{ of Type 1} \\ K \le Y_0}} \frac{1}{K^{1/2}} \prod_{p \in \mathcal{T}_K} \left(1 + \frac{1}{p}\right)^{-1}$. It is obvious that $\lim_{Y_0 \to \infty} D(Y_0) = D \ (< +\infty)$ exists.

We may then complete the proof of the Theorem by showing that, letting $d_5 = d_4 D$,

$$E(x) = (1 + o(1)) d_5 \sqrt{\frac{x}{\log x}}. \tag{20}$$

Indeed, from (19), we have

$$\left|E(x) - d_5\sqrt{\frac{x}{\log x}}\right| \le d_4|D - D(Y_0)|\sqrt{\frac{x}{\log x}} + O\left(\delta(Y_0)\sqrt{\frac{x}{\log x}}\right) + o\left(\sqrt{\frac{x}{\log x}}\right),$$

thus implying that, for some positive constant d_6,

$$\left|\frac{E(x)}{\sqrt{\frac{x}{\log x}}} - d_5\right| \le d_4|D - D(Y_0)| + d_6\delta(Y_0) + o(1).$$

Thus, the relation

$$\limsup_{x\to\infty}\left|\frac{E(x)}{\sqrt{\frac{x}{\log x}}} - d_5\right| \le d_4|D - D(Y_0)| + d_6\delta(Y_0)$$

holds for every large number Y_0, which means that it holds for $Y_0 \to \infty$. Hence, using the fact that $D(Y_0) \to D$ and that $\delta(Y_0) \to 0$, it follows that

$$\limsup_{x\to\infty}\left|\frac{E(x)}{\sqrt{\frac{x}{\log x}}} - d_5\right| = 0,$$

which proves (20), thus completing the proof of the Theorem.

References

[1] P.T. Bateman and H.G. Diamond, *Analytic Number Theory*, World Scientific, New Jersey, 2004.

[2] H.J. Kanold, *Über das harmonische Mittel der Teiler einer natürlichen Zahl II*, Math. Annalen **134** (1958), 225-231.

JMDK, le 6 janvier 2006; fichier: kanold2006.tex