Ann. Sci. Math. Québec **30** (2006), no 1, 55–62

A CONSEQUENCE OF A THEOREM OF FILASETA

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RÉSUMÉ. Soit $\tau^{(e)}(n)$ le nombre de diviseurs exponentiels de n. En nous appuyant sur un théorème de Filaseta portant sur le nombre d'entiers libres de puissances k-ièmes dans l'intervalle $(x, x+h(x)]$, où $h(x) = x^{1/(2k+1)} g(x)^3$ avec $1 \le g(x) \le \log x$ pour x suffisamment grand, nous démontrons que si $h(x) = x^{1/5}(\log x)^{3/2} u(x)$ où $h(x) \le x$ et $u(x) \to \infty$ lorsque $x \to \infty$, alors pour tout nombre réel $\alpha > 0$ donné, il existe une constante $c = c(\alpha) > 0$ telle que

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} \tau^{(e)}(n)^\alpha = c.$$

ABSTRACT. Let $\tau^{(e)}(n)$ be the number of exponential divisors of n. Using a theorem of Filaseta on the number of k-free integers belonging to the interval $(x, x+h(x)]$, where $h(x) = x^{1/(2k+1)} g(x)^3$ with $1 \le g(x) \le \log x$ for x sufficiently large, we prove that if $h(x) = x^{1/5}(\log x)^{3/2} u(x)$ where $h(x) \le x$ and $u(x) \to \infty$ as $x \to \infty$, then for each fixed real number $\alpha > 0$, there exists a constant $c = c(\alpha) > 0$ such that

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} \tau^{(e)}(n)^\alpha = c.$$

1. Introduction. Let $\mathscr{P}$ be the set of prime numbers, $\mathscr{B}$ the set of squarefull numbers and $\mathscr{M}$ the set of squarefree numbers. Let also $\tau(n)$ stand for the number of positive divisors of n. Finally, let $\tau^{(e)}(n)$ be the number of exponential divisors of n, i.e., $\tau^{(e)}(n)$ is the multiplicative function defined on the prime powers p^α by $\tau^{(e)}(p^\alpha) = \tau(\alpha)$. The notion of exponential divisors was first introduced by Subbarao [4].

Recently, Kátai and Subbarao [3], among others, investigated the expression

$$S(x,H) = \frac{1}{H} \sum_{x\le n\le x+H} \tau^{(e)}(n).$$

They proved that there exists a positive constant c such that $S(x,H) = c + o(1)$ as $x \to \infty$, provided that $H = H(x) = x^{\theta+\varepsilon}$, where $\theta = 0.2204$ and ε is any given positive real number. The main ingredient was a result of Varbanec [5] quoted below.

Lemma 1 (VARBANEC). *Let $\delta > 0$ be an arbitrary fixed number. Let $\phi(d)$ be a multiplicative function such that $\phi(d) = O(d^\delta)$ and let $f(n) = \sum_{d^2|n} \phi(d)$. Then the*

Reçu le 2 mars 2005 et, sous forme définitive, le 14 août 2006.

*Professor M. V. Subbarao passed away on February 15, 2006.

estimate

$$\sum_{x\le n\le x+h} f(n) = h\sum_{d=1}^{\infty}\frac{\phi(d)}{d^2} + O(h^{1/2}x^{\delta}) + O(x^{\theta+\delta})$$

holds uniformly for $h<x$, where $\theta = 0.2204$.

We will also use the following result.

Lemma 2. *Given an integer $k\ge 2$, set $h(x) = x^{1/(2k+1)}\log^3 x$ and let $E(x)$ be the number of integers $n\in[x,x+h(x)]$ that can be written as $n = p^k\nu$ for some prime power $p^k \ge h(x)$. Then*

$$E(x) \ll \frac{h(x)}{\log x}.$$

Lemma 2 is an immediate consequence of Theorem A (stated below). The latter is an unpublished result of Michael Filaseta who kindly communicated its statement and proof to the first author. A complete proof due to Filaseta is given by De Koninck and Kátai [1]; see also Filaseta and Trifonov [2].

Theorem A (FILASETA). *Given an integer $k\ge 2$, let $g(x)$ be a function satisfying $1\le g(x)\le \log x$ for x sufficiently large, and set*

$$h(x) = x^{1/(2k+1)}g(x)^3.$$

Then the number of k-free numbers belonging to the interval $(x,x+h(x)]$ is

$$\frac{h(x)}{\zeta(k)} + O\left(\frac{h(x)\cdot\log x}{g(x)^3}\right) + O\left(\frac{h(x)}{g(x)}\right),$$

where ζ stands for the Riemann Zeta Function.

2. Consequence of the theorem of Filaseta. The following result is an important consequence of the theorem of Filaseta.

Theorem 1. *Let $h(x) = x^{1/5}(\log x)^{3/2}u(x)$, where $h(x)\le x$ and $u(x)\to\infty$ as $x\to\infty$. Let also $\alpha>0$ be a fixed real number. Then, as $x\to\infty$,*

$$\frac{1}{h(x)}\sum_{n\in[x,x+h(x)]}\tau^{(e)}(n)^{\alpha} = c + o(1),$$

where $c = c(\alpha)$ is a positive constant given by (10).

Proof. It is enough to prove the theorem for a function $u(x)$ which tends to infinity sufficiently slowly. Let

$$M(x) = \sum_{n\le x}|\mu(n)|$$

and for each integer $k\ge 2$, let

$$M(x|k) = \sum_{\substack{n\le x\\(n,k)=1}}|\mu(n)|.$$

Also, for each integer $m\ge 2$, let $\mathscr{D}_m$ be the multiplicative semigroup generated by the prime powers dividing m, i.e., let

$$\mathscr{D}_m = \left\{p_1^{\beta_1}\dots p_r^{\beta_r} : \beta_i\in\mathbb{N}_0\right\}$$

if $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ (with each $\alpha_i \in \mathbb{N}$), the prime factorization of m, where $\mathbb{N}$ (resp. $\mathbb{N}_0$) stands for the set of positive integers (resp. non-negative integers).

Now let λ be the Liouville function defined by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ stands for the number of prime factors of n counted with their multiplicity. Starting from the Dirichlet generating functions identity

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{\infty} \frac{|\mu(n)|}{n^s} = \sum_{\substack{v=1 \\ v \in \mathcal{D}_k}}^{\infty} \frac{\lambda(v)}{v^s} \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s},$$

we obtain

$$\begin{aligned} M(y+H|k) \quad - \quad M(y|k) &= \sum_{\substack{y<n\le y+H \\ (n,k)=1}} |\mu(n)| \qquad (1) \\ &= \sum_{v \in \mathcal{D}_k} \lambda(v) \sum_{\frac{y}{v}<m\le \frac{y+H}{v}} |\mu(m)| \\ &= \sum_{v \in \mathcal{D}_k} \lambda(v) \left\{ M\left(\frac{y+H}{v}\right) - M\left(\frac{y}{v}\right) \right\}. \end{aligned}$$

From here on, we shall write each integer n as the product of its squarefull part and its squarefree part, namely $n = km$, where $k \in \mathcal{B}$, $m \in \mathcal{M}$ and $(m,k) = 1$. For short, we write $f(n)$ for $\tau^{(e)}(n)^{\alpha}$, observing by the way that $f(m) = 1$ for each $m \in \mathcal{M}$. Let $v(x)$ be a function tending to infinity as $x \to \infty$, but slowly enough that

$$\prod_{p \le v(x)} \left(\frac{\log(2x)}{\log p} + 1 \right) \le x^{1/10}.$$

Moreover, let $T_k = \#\{v \in \mathcal{D}_k : v \le 2x\}$. Then we have

$$T_k \le x^{1/10} \quad \text{provided } k \le v(x). \qquad (2)$$

Now let

$$\sum_{x \le n \le x+h(x)} f(n) = \Sigma_1 + \Sigma_2, \qquad (3)$$

where the sum in Σ_1 is over those integers $n \in [x, x+h(x)]$ for which $k < v(x)$, while the sum in Σ_2 is over the others.

First observe that

$$\Sigma_1 = \sum_{\substack{k<v(x) \\ k \in \mathcal{B}}} f(k) \left\{ M\left(\frac{x+h(x)}{k} \Big| k \right) - M\left(\frac{x}{k} \Big| k \right) \right\}. \qquad (4)$$

Using (1) and Theorem A, we find

$$\begin{aligned} M\left(\frac{x+h(x)}{k} \Big| k \right) \quad - \quad M\left(\frac{x}{k} \Big| k \right) & \qquad (5) \\ = \sum_{\substack{v \in \mathcal{D}_k \\ v < v(x)}} \lambda(v) \left\{ \frac{6}{\pi^2} \frac{h(x)}{kv} + O\left(\frac{h(x)}{kv \cdot (\log x)^{1/2}} \right) \right\} & \\ + O\left(\frac{h(x)}{k} \sum_{\substack{v \ge v(x) \\ v \in \mathcal{D}_k}} \frac{1}{v} \right) + O(S_k), & \end{aligned}$$

where

$$S_k = \sum_{\substack{v\in\mathscr{D}_k \\ x\le vkm\le x+h(x) \\ vk>h(x)}} 1.$$

Observe that in the sum defining S_k, for each v, there exists at most one m such that $vkm \in [x, x+h(x)]$. It thus follows, in the light of (2), that $S_k \le T_k \le x^{1/10}$.

Now note that

$$\sum_{v\in\mathscr{D}_k} \frac{1}{\sqrt{v}} \le \prod_{p|k}\left(1+\frac{1}{\sqrt{p}}+\frac{1}{p}+\cdots\right) \le \exp\left(\sum_{p|k}\frac{2}{\sqrt{p}}\right).$$

Consequently, the first error term on the last line of (5) is

$$\ll \frac{h(x)}{k\sqrt{v(x)}}\sum_{v\in\mathscr{D}_k}\frac{1}{\sqrt{v}} \ll \frac{h(x)}{k\sqrt{v(x)}}\exp\left(\sum_{p|k}\frac{2}{\sqrt{p}}\right). \tag{6}$$

Hence, taking into account that

$$\sum_{\substack{v\ge v(x) \\ v\in\mathscr{D}_k}} \frac{1}{kv} \ll \frac{1}{k\sqrt{v(x)}}\exp\left(\sum_{p|k}\frac{2}{\sqrt{p}}\right),$$

$$\sum_{v\in\mathscr{D}_k}\frac{1}{v} \ll \prod_{p|k}\left(1+\frac{1}{p}\right) \quad \text{and} \quad \sum_{v\in\mathscr{D}_k}\frac{\lambda(v)}{v} = \prod_{p|k}\frac{1}{1+1/p},$$

we find, upon substituting (6) in (5), that

$$\begin{aligned} M\left(\frac{x+h(x)}{k}\Big| k\right) &- M\left(\frac{x}{k}\Big| k\right) \qquad (7)\\ &= \frac{6}{\pi^2}\frac{h(x)}{k}\prod_{p|k}\frac{1}{1+1/p} + O\left(\frac{h(x)}{k\sqrt{v(x)}}\exp\left(\sum_{p|k}\frac{2}{\sqrt{p}}\right)\right)\\ &\quad + O\left(\frac{h(x)}{k\sqrt{\log x}}\prod_{p|k}\left(1+\frac{1}{p}\right)\right),\end{aligned}$$

because T_k is no larger than the first of the above error terms.

Thus, from (4) and (7), we see that

$$\begin{aligned}\Sigma_1 &= \frac{6}{\pi^2}h(x)\Sigma_0 + O\left(\frac{h(x)}{\sqrt{v(x)}}\sum_{k\in\mathscr{B}}\frac{f(k)}{k}\exp\left(\sum_{p|k}\frac{2}{\sqrt{p}}\right)\right) \qquad (8)\\ &\quad + O\left(\frac{h(x)}{\sqrt{\log x}}\sum_{k\in\mathscr{B}}\frac{f(k)}{k}\prod_{p|k}\left(1+\frac{1}{p}\right)\right),\end{aligned}$$

where

$$\Sigma_0 = \Sigma_0(x) = \sum_{\substack{k<v(x) \\ k\in\mathscr{B}}}\frac{f(k)}{k}\prod_{p|k}\frac{1}{1+1/p}. \tag{9}$$

Clearly, the series on the right-hand side of (9) is convergent when extended to all $k\in\mathscr{B}$. It follows that

$$\Sigma_0 = c + o(1),$$

where

$$c = c(\alpha) = \sum_{k\in\mathscr{B}} \frac{f(k)}{k} \prod_{p|k} \frac{1}{1+1/p} = \sum_{k\in\mathscr{B}} \frac{\tau^{(e)}(k)^{\alpha}}{k} \prod_{p|k} \frac{1}{1+1/p}. \tag{10}$$

Moreover, given that both series appearing in the error terms on the right-hand side of (8) are convergent when k runs over all numbers in $\mathscr{B}$, one can see that (8) can be written as

$$\Sigma_1 = \frac{6}{\pi^2} h(x)\, c(\alpha) + o(h(x)). \tag{11}$$

It remains to estimate Σ_2. In order to do so, we split it into two sums, namely

$$\Sigma_2 = \Sigma_{2,1} + \Sigma_{2,2}, \tag{12}$$

where the sum in $\Sigma_{2,1}$ is over those integers $n \in [x, x+h(x)]$ whose squarefull part k belongs to $[v(x), h(x)]$, while the sum in $\Sigma_{2,2}$ is over those $n \in [x, x+h(x)]$ such that $k > h(x)$. It is clear that

$$\Sigma_{2,1} \ll h(x) \sum_{v(x)\le k\le h(x)} \frac{f(k)}{k} = o(h(x)). \tag{13}$$

Let us now consider $\Sigma_{2,2}$. For this, let

$$\mathscr{J} = \{k : k \in \mathscr{B},\ k > h(x),\ k\nu \in [x, x+h(x)] \text{ for some } \nu\},$$

which allows us to write

$$\Sigma_{2,2} \le \sum_{k\in\mathscr{J}} f(k).$$

Let $\varepsilon > 0$ be a fixed small number, and let $\mathscr{J}_1$ be the set of those $k \in \mathscr{J}$ which have a squarefull divisor k_1 such that $k_1 \in [x^{\varepsilon}, x^{1/5}]$. Let also

$$\Sigma_{2,2,1} = \sum_{k\in\mathscr{J}_1} f(k) \quad \text{and} \quad \Sigma_{2,2,2} = \sum_{k\in\mathscr{J}_2} f(k),$$

where $\mathscr{J}_2 = \mathscr{J} \setminus \mathscr{J}_1$. Observe that $f(m) \le Cm^{\varepsilon/4}$ say, and hence that

$$\Sigma_{2,2,1} \le 2Cx^{\varepsilon/4} \sum_{\substack{k_1 > x^{\varepsilon} \\ k_1 \in\mathscr{B}}} \frac{h(x)}{k_1} \le 2Ch(x)x^{\frac{\varepsilon}{4}-\frac{\varepsilon}{2}} \ll h(x)x^{-\varepsilon/4}. \tag{14}$$

Now if $k \in \mathscr{J}_2$, then $k = ab$, where $a, b \in \mathscr{B}$, $a < x^{\varepsilon}$ and $p(b) > \sqrt{h(x)}$, where $p(b)$ stands for the smallest prime factor of b. It is clear that $f(b) = O(1)$, so that $f(k) \ll f(a)$. Hence, letting Y be a fixed large number, we may write

$$\Sigma_{2,2,2,1} = \sum_{\substack{k\in\mathscr{J}_2 \\ a>Y,\ a\in\mathscr{B}}} f(k) \ll h(x) \sum_{\substack{a>Y \\ a\in\mathscr{B}}} \frac{f(a)}{a} \ll \frac{h(x)}{Y^{1/2-\varepsilon}}. \tag{15}$$

It remains to estimate $\Sigma_{2,2,2,2}$, i.e., the sum of $f(k)$ running over those $k \in \mathscr{J}_2$ for which $a \le Y$. Recalling that $f(k) \ll f(a)$ and letting $T(Y) = \max_{a\le Y} f(a)$, we have

$$\Sigma_{2,2,2,2} \ll T(Y) \cdot \#\{p^2\nu \in [x, x+h(x)] : p \text{ prime},\ p > \sqrt{h(x)}\}.$$

Thus, by Lemma 2, we have

$$\Sigma_{2,2,2,2} \ll \frac{T(Y)}{\log x} \cdot h(x). \tag{16}$$

Choosing $Y = \log x$ and observing that $T(Y) \ll Y^{\varepsilon}$, we conclude from (16) that $\Sigma_{2,2,2,2} = o(h(x))$. Combined with (15) and in the light of (14), this yields $\Sigma_{2,2} = o(h(x))$. Therefore, substituting this result and the estimate (13) in (12), we obtain $\Sigma_2 = o(h(x))$. Substitution of this estimate along with (11) in (3) completes the proof of the theorem. □

Remark. Without any major modification, our method can be applied to a broader class of multiplicative functions. Namely, one can generalize Theorem 1 as follows.

Theorem 2. Let f be a non-negative multiplicative function such that $f(p) = 1$ for each prime p and $f(n) = O(n^{\varepsilon})$ for every $\varepsilon > 0$. Let also $h(x) = x^{1/5}(\log x)^{3/2}u(x)$, where $u(x) \to \infty$ as $x \to \infty$, with $h(x) \leq x$. Then

$$\frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} f(n) = c + o(1),$$

where $c = c(f)$ is a positive constant given by

$$c(f) = \sum_{k\in\mathscr{B}} \frac{f(k)}{k} \prod_{p|k} \left(1+\frac{1}{p}\right)^{-1}.$$

Acknowledgements. The authors are grateful to a referee for comments that led to an improved version of this paper. Funding in partial support of the first author's work was provided by the Natural Sciences and Engineering Research Council of Canada. The second author's research was supported by the Applied Number Theory Research Group of the Hungarian Academy of Science and by a grant from OTKA.

Résumé substantiel en français. Soit f une fonction multiplicative non négative telle que $f(p) = 1$ pour chaque nombre premier p et telle que $f(n) = O(n^{\varepsilon})$ pour chaque $\varepsilon > 0$. Soit également $h(x) = x^{1/5}(\log x)^{3/2}u(x)$, où $u(x) \to \infty$ lorsque $x \to \infty$, avec $h(x) \leq x$. Les auteurs démontrent que

$$\frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} f(n) = c + o(1),$$

où $c = c(f)$ est une constante positive définie par

$$c(f) = \sum_{k\in\mathscr{B}} \frac{f(k)}{k} \prod_{p|k} \left(1+\frac{1}{p}\right)^{-1},$$

où $\mathscr{B}$ désigne l'ensemble des nombres puissants, soit les entiers positifs n tels que $p|n \Rightarrow p^2|n$.

La motivation des auteurs est en réalité le cas particulier $f(n) = \tau^{(e)}(n)$, qui désigne le nombre de diviseurs exponentiels de n, soit la fonction multiplicative définie sur les puissances de nombres premiers p^{α} par $\tau^{(e)}(p^{\alpha}) = \tau(\alpha)$, où $\tau(\alpha)$ désigne le nombre de diviseurs de α. Les auteurs obtiennent ainsi l'estimation

$$\frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} \tau^{(e)}(n)^{\alpha} = c + o(1),$$

où

$$c = \sum_{k\in\mathscr{B}} \frac{\tau^{(e)}(k)^\alpha}{k} \prod_{p|k} \frac{1}{1+1/p},$$

ce qui améliore un résultat obtenu en 2003 par Kátai et Subbarao, à savoir l'estimation

$$\frac{1}{H} \sum_{x\le n\le x+H} \tau^{(e)}(n) = c + o(1),$$

où $H = H(x) = x^{\theta+\varepsilon}$ dans laquelle θ = 0,2204 et ε est un nombre réel positif arbitrairement petit.

L'obtention du résultat général ci-dessus repose en grande partie sur un théorème récent de Michael Filaseta, à savoir : *Étant donné un entier $k \ge 2$, soit $g(x)$ une fonction satisfaisant $1 \le g(x) \le \log x$ pour x suffisamment grand, et soit*

$$h(x) = x^{1/(2k+1)} g(x)^3.$$

Alors le nombre d'entiers libres de puissances k-ièmes appartenant à l'intervalle $(x, x+h(x)]$ est

$$\frac{h(x)}{\zeta(k)} + O\left(\frac{h(x)\cdot\log x}{g(x)^3}\right) + O\left(\frac{h(x)}{g(x)}\right),$$

où ζ désigne la fonction zêta de Riemann.

References

1. J.-M. De Koninck and I. Kátai, *On the mean value of the index of composition*, Monatshefte für Mathematik **145** (2005), 131–144.
2. M. Filaseta and O. Trifonov, *The distribution of fractional parts with applications to gap results in number theory*, Proc. London Math. Soc. **73** (1996), 241–278.
3. I. Kátai and M. V. Subbarao, *On the distribution of exponential divisors*, Ann. Univ. Sci. Budapest, Sect. Comp. **22** (2003), 161–180.
4. M. V. Subbarao, *On some arithmetic convolutions*, The Theory of Arithmetic Functions, Lecture Notes in Mathematics **251**, Springer, 1972, pp. 247–271.
5. P. Varbanec, *Multiplicative functions of special type in short intervals*, New trends in Probability and Statistics, Vol. 2, Analytic and Probabilistic Methods in Number Theory, TEV, Vilnius, 1992, pp. 181–188.

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