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A SCHINZEL HYPOTHESIS H TYPE OF RESULT FOR ECONOMICAL NUMBERS

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RÉSUMÉ. Nous établissons que si $f_1(X)$, $f_2(X)$, ... , $f_t(X)$ sont des polynômes non constants à coefficients entiers, alors il existe une infinité d'entiers positifs n tels que $f_i(n)$ est un nombre économique pour $i = 1, \ldots, t$.

ABSTRACT. We show that if $f_1(X)$, $f_2(X)$, ... , $f_t(X)$ are nonconstant polynomials with integer coefficients, then there exist infinitely many positive integers n such that $f_i(n)$ is an economical number for $i = 1, \ldots, t$.

1. Introduction. In 1995, Bernardo Recamán Santos [6] defined a number n to be *equidigital* if the prime factorization of n requires the same number of decimal digits as n, and *economical* if its prime factorization requires no more digits. He asked whether there are arbitrarily long sequences of consecutive economical numbers. In 1998, Richard Pinch [5] gave an affirmative answer to this question assuming the *prime k-tuples conjecture* stated by L. E. Dickson [1] in 1904. He also exhibited one such sequence of length nine starting with the 19-digit number 1034429177995381247 and conjectured that such a sequence of arbitrary length always exists. In [2], the first two authors proved that Pinch's result holds unconditionally. Upper and lower bounds for the counting function of the set of economical numbers, i.e. for the cardinality of the set $\{n \le x \,:\, n \text{ economical}\}$ can be found in [3].

In this paper, we extend the result from [2]. We recall that Schinzel's Hypothesis H is the following conjecture: *Assume that $f_1(X), \ldots, f_t(X) \in \mathbb{Q}[X]$ are irreducible polynomials which are integer valued for $n \in \mathbb{Z}$ of degrees > 1 with positive leading terms; assume also that there does not exist any prime number p dividing $f_1(n) \cdots f_t(n)$ for all $n \in \mathbb{Z}$. Then there exist infinitely many positive integers n such that $f_i(n)$ are primes for $i = 1, \ldots, t$.* In this paper, we prove that a result of this type holds if we replace the requirement that $f_i(n)$ are primes for $i = 1, \ldots, t$ by the requirement that $f_i(n)$ are economical for $i = 1, \ldots, t$.

2. Main result. In what follows, we write $B > 1$ for a positive integer and $\mathcal{E}_B$ for the set of strongly economical numbers in base B, namely those numbers n whose prime factorization in base B (including the exponents > 1) requires less digits than n. Our main result is the following:

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Theorem 2.1. *Let $f_1(X),\ f_2(X),\ \dots,\ f_t(X) \in \mathbb{Z}[X]$ be nonconstant polynomials. Then there exist infinitely many positive integers n such that all the numbers $|f_1(n)|,\ \dots,\ |f_t(n)|$ are in $\mathcal{E}_B$.*

Let $S_B(n)$ stand for the number of digits of n in base B and $T_B(n)$ for the number of digits of the prime factorization of n in base B. In his 1998 paper [5], Pinch obtained several elementary properties regarding these functions, two of which we state in the following lemma:

Lemma 2.2. *For all positive integers m, n,*

$$S_B(m) + S_B(n) \le S_B(mn) + 1 \qquad \textit{and} \qquad T_B(mn) \le T_B(m) + T_B(n).$$

We shall also need following lemma.

Lemma 2.3. *If $m,\ n \in \mathcal{E}_B$, then $mn \in \mathcal{E}_B$.*

Proof. Using Lemma 2.2, if $m,\ n \in \mathcal{E}_B$, then $T_B(m) \le S_B(m) - 1$ and $T_B(n) \le S_B(n) - 1$, so that $T_B(mn) \le T_B(m) + T_B(n) \le (S_B(m) + S_B(n) - 1) - 1 \le S_B(mn) - 1$, showing that $mn \in \mathcal{E}_B$. $\square$

In [2], the first two authors obtained a simple criterion for an integer n to be in $\mathcal{E}_B$. This result is contained in the following lemma, of which we also give a proof for the sake of completeness.

Lemma 2.4. *Let n be a positive integer. Assume that there exist a prime q and a positive integer β such that $q^\beta | n$ with*

$$q^{\beta-1} > B^{4\omega(n)+2}. \tag{2.1}$$

Then $n \in \mathcal{E}_B$.

Proof. First of all, it is clear that for each integer $n = \prod_{p^{\alpha_p}\|n} p^{\alpha_p} \geqslant 2$, we have

(i) $S_B(n) - T_B(n) > \dfrac{1}{\log B} \cdot \log\left(\prod_{p^{\alpha_p}\|n} \dfrac{p^{\alpha_p - 1}}{\alpha_p}\right) - 2\omega(n),$

(ii) $\dfrac{p^{\alpha_p-1}}{\alpha_p} \geqslant 1 \qquad (p \geqslant 2,\ \alpha_p \geqslant 1),$

(iii) $\dfrac{p^{\alpha_p-1}}{\alpha_p} \geqslant \dfrac{1}{2} p^{\frac{\alpha_p - 1}{2}} \qquad (p \geqslant 2,\ \alpha_p \geqslant 1).$

Note that (i) follows from the two inequalities

$$S_B(n) > \frac{\log n}{\log B} \quad \text{and} \quad T_B(n) < 2\omega(n) + \sum_{p^{\alpha_p}\|n} \frac{\log(p\alpha_p)}{\log B}.$$

Now, using (ii) and (iii), and then (2.1), we get

$$\prod_{p^{\alpha_p}\|n} \frac{p^{\alpha_p-1}}{\alpha_p} \geqslant \frac{q^{\beta-1}}{\beta} > \frac{q^{(\beta-1)/2}}{2} \geqslant \frac{B^{2\omega(n)+1}}{2} \geqslant B^{2\omega(n)}.$$

Hence, using (i), we obtain that

$$S_B(n) - T_B(n) > \frac{1}{\log B} \cdot \log B^{2\omega(n)} - 2\omega(n) = 0,$$

thus completing the proof of the lemma. □

We are now ready to prove Theorem 2.1.

Proof. In light of Lemma 2.3, we may assume that all polynomials $f_i(X)$ are irreducible, for if not we may decompose them in irreducible factors and prove the theorem for the set of polynomials $\{f_i(X) : i = 1, \dots, t\}$ replaced by the set of irreducible factors of those. Let A be larger than the maximum between the absolute values of all the coefficients and discriminants of $f_i(X)$ for $i = 1, \dots, t$. Assume further that $A > t$. Let $p_1, \dots, p_t$ be distinct primes such that $p_i > A$ for all $i = 1, \dots, t$ and the equation $f_i(n) \equiv 0 \pmod{p_i}$ is solvable in $\mathbb{Z}$ for $i = 1, \dots, t$. For the existence of such primes, see, for example, Exercise 1.2.5 on page 7 in [4]. Let $a_i \in \mathbb{Z}$ be such that if $n \equiv a_i \pmod{p_i}$, then $f_i(n) \equiv 0 \pmod{p_i}$. Now let x be a large positive real number, and let $y = \lfloor \log x / \log\log\log x \rfloor$. Now a_i is a simple root of $f_i(X)$ modulo p_i, because $p_i > A$ is larger than the absolute value of the discriminant of $f_i(X)$ and $f_i(X)$ is irreducible, in particular, it does not have double roots. It follows, by Hensel's Lemma, that there exists $a_i(y) \in [0, \dots, p_i^y - 1]$ such that if $n \equiv a_i(y) \pmod{p_i^y}$, then $f_i(n) \equiv 0 \pmod{p_i^y}$. By the Chinese Remainder Theorem, there exists an integer $a(y)$ in $[0, \dots, M-1]$, where $M = (\prod_{i=1}^t p_i)^y$, such that if $n \equiv a(y) \pmod{M}$, then $f_i(n) \equiv 0 \pmod{p_i^y}$ for $i = 1, \dots, t$. Note that $M = \exp(O(y)) = \exp(O(\log x / \log\log\log x))$. The number of such positive integers $n \le x$ is $\ge \lfloor x/M \rfloor = x^{1+o(1)}$. Let $n \le x$ be such an integer. Let D be an upper bound for the degrees of all the polynomials $f_i(X)$ for $i = 1, \dots, t$. Then $|f_i(n)| \ll x^D$. Since the inequality $\omega(m) \ll \log m / \log\log m$ holds for all positive integers m, we get that $\omega(|f_i(n)|) \ll \log x / \log\log x$. Thus,

$$B^{4\omega(|f_i(n)|)+2} = \exp(O(\log x / \log\log x)).$$

Since $y - 1 \gg \log x / \log\log\log x$, it follows easily that if x is large, then the inequality

$$p_i^{y-1} > B^{4\omega(|f_i(n)|)+2}$$

holds for all $i = 1, \dots, t$ and all $n \equiv a(y) \pmod{M}$. Since $p_i^y \| |f_i(n)|$, by Lemma 2.4, we get that $f_i(n) \in \mathcal{E}_B$, which concludes the proof of Theorem 2.1. □

Remark. Taking $f_i(n) = n + i$, we get an alternative (and easier) proof of the fact that there exist arbitrarily large strings of consecutive economical numbers. We do point out however that the result from [2] is effective in the sense that the main result there is that there exists some constant c depending on B such that for infinitely many positive integers n all integers $n + j$ for $j = 1, \dots, \lfloor c \log\log n \rfloor$ are in $\mathcal{E}_B$.

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Résumé substantiel en français. En 1995, Santos a introduit la notion de *nombre économique*, soit un entier positif n dont la factorisation ne nécessite pas plus de chiffres que sa représentation décimale. Ainsi $125 = 5^3$ est un nombre économique, alors que $40 = 2^3 \cdot 5$ ne l'est pas. Santos avait demandé s'il existait des suites arbitrairement longues de nombres consécutifs tous économiques. En 1998, Pinch répondait à cette question dans l'affirmative, mais en supposant une vieille conjecture de Dickson appelée *la conjecture des nombres premiers jumeaux généralisée*. En 2003, De Koninck et Luca démontraient le résultat de Pinch, mais sans aucune condition. Récemment, les deux mêmes auteurs établissaient des bornes inférieure et supérieure pour la quantité de nombres économiques n'excédant pas un nombre donné x. Dans le présent article, nous démontrons que si $f_1(X), f_2(X), \ldots, f_t(X)$ sont des polynômes non constants à coefficients entiers, alors il existe une infinité d'entiers positifs n tels que chacun des nombres $|f_1(n)|, |f_2(n)|, \ldots, |f_t(n)|$ est un nombre économique.

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