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COMPUTATIONAL RESULTS AND QUERIES IN NUMBER THEORY

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Dedicated to Professor Imre Kátai on his 65-th anniversary

1. Introduction

According to Moore's Law, the speed of modern computers doubles every 18 months. Hence it is no surprise that several old and new problems in number theory have gained special interest, mainly because one can study them using both a theoretical approach and a computational one. In this paper, we examine two such challenging problems.

Throughout this paper, we denote by $\mathbf{N}$ the set of positive integers and by $\mathbf{P}$ the set of prime numbers. Furthermore, given an integer $n \geq 2$, we shall write $P(n)$ to denote its largest prime factor and $\omega(n)$ to denote the number of distinct prime factors of n.

2. Sum of the digits of prime numbers

Given a positive integer n, let $s(n)$ stand for the sum of its (decimal) digits. Given a positive integer k which is not a multiple of 3, let $\rho(k)$ be the smallest prime number p such that $s(p) = k$. Below, is the table of values of $\rho(k)$ for $k = 2, 4, 5, 7, 8, \ldots, 83$.

Is it clear that $\rho(k)$ is a well defined function? Not at all! In fact, it is an open problem. Nevertheless, given any integer $k \geq 4$ which is not a multiple of 3, the set $A_k := \{p \in \mathbf{P} : \ s(p) = k\}$ is expected to be non empty and, in fact, is most likely infinite.

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k	$\rho(k)$
2	2
4	13
5	5
7	7
8	17
10	19
11	29
13	67
14	59
16	79
17	89

k	$\rho(k)$
19	199
20	389
22	499
23	599
25	997
26	1889
28	1999
29	2999
31	4999
32	6899
34	17989

k	$\rho(k)$
35	8999
37	29989
38	39989
40	49999
41	59999
43	79999
44	98999
46	199999
47	389999
49	598999
50	599999

k	$\rho(k)$
52	799999
53	989999
55	2998999
56	2999999
58	4999999
59	6999899
61	8989999
62	9899999
64	19999999
65	29999999
67	59899999

k	$\rho(k)$
68	59999999
70	189997999
71	89999999
73	289999999
74	389999999
76	689899999
77	699899999
79	799999999
80	998999999
82	2999899999
83	3999998999

The size of A_2 is a different matter. Indeed, since one can easily show that any element of A_2 larger than 2 must be of the form $a_r := 10^{2^r} + 1$ and since a_0 and a_1 are primes, and since, using a computer, one can establish that a_r is composite for $2 \le r \le 17$, it follows that any prime number $p \in A_2$, $p \ne$ $\ne 2, 11, 101$, must be larger than $2^{2^{18}}$. Therefore, it is possible that $\#A_2 = 3$.

On the other hand, one can easily show that $\#A_k \ge 1$ infinitely often. Indeed, assume the contrary, meaning that $\#A_k \ge 1$ only a finite number of times, and therefore assume that there exists a positive integer k_0 such that $\#A_k = 0$ for all $k \ge k_0$, thus implying that $s(p) < k_0$ for all primes p. Let $\ell = [k_0/9] + 1$ and consider the sequence of integers

$$a_m := 10^\ell \cdot m + \underbrace{99\ldots9}_{\ell \text{ times}} \quad (m = 1, 2, \ldots).$$

It follows from Dirichlet's theorem that there exists a positive integer m_0 such that a_{m_0} is a prime number, in which case

$$s(a_{m_0}) = s(m_0) + 9\ell > s(m_0) + 9\frac{k_0}{9} \ge 1 + k_0 > k_0,$$

thus contradicting the fact that $s(p) < k_0$ for all primes p.

Finally, observe that $\rho(k)$ cannot be "too large" all the time; in fact, one can show that there exist infinitely many positive integers k such that $\rho(k) \le 10^{2k}$. Indeed, assume that there are only finitely many positive integers k such that $\rho(k) \le 10^{2k}$, in which case there exists an integer k_0 such that $\rho(k) > 10^{2k}$ for all $k \ge k_0$. This means in particular that, for all $k \ge k_0$, each prime p with $s(p) = k$ is such that $p > 10^{2k}$. Therefore, in particular, if $p \le 10^{2k_0}$ then $s(p) < k_0$. But this is impossible since the number of primes $p \le 10^{2k_0}$ is much larger than $S(k_0)$, the number of primes p with no more than $2k_0$ digits and such that $s(p) < k_0$. Indeed,

(1)

$$\begin{aligned} S(k_0) &< \sum_{\substack{k=2 \\ k/3 \notin N}}^{k_0-1} \#\{p \in \mathbf{P} : s(p) = k \text{ with } p \text{ having no more than } 2k_0 \text{ digits}\} < \\ &< \sum_{k=2}^{k_0-1} \#\{n \in \mathbf{N} : s(n) = k \text{ with } n \text{ having no more than } 2k_0 \text{ digits}\} < \\ &< \sum_{k=2}^{k_0-1} p_9(k, 2k_0 - 1), \end{aligned}$$

where $p_r(k, 2k_0 - 1)$ stands for the number of partitions of k into parts $\in [0, r]$, but with no more than $2k_0 - 1$ zeros amongst its parts. Since $p_9(k, 2k_0 - 1) <$ $< p_9(k) \cdot 2^{2k_0 - 1}$, where $p_r(k)$ is the number of partitions of k into parts $\in [1, r]$, then it follows from (1) that

$$S(k_0) < 4^{k_0} \sum_{k=2}^{k_0-1} p_9(k). \tag{2}$$

Using the general estimate

$$p_r(k) \sim \frac{k^{r-1}}{r!(r-1)!} \quad (k \to \infty)$$

(see Corollary 15.1 of Nathanson [4]), it follows from (2) that

$$S(k_0) \ll 4^{k_0} \sum_{k=2}^{k_0-1} \frac{k^8}{9!8!} \ll 4^{k_0} k_0^9, \tag{3}$$

a quantity which is clearly smaller that

$$\pi(10^{2k_0}) \approx \frac{10^{2k_0}}{2k_0 \log 10}$$

if k_0 is sufficiently large, thus proving our claim.

Here are some other open problems concerning $\rho(k)$:

1. It is easy to show that if $\rho(k)$ exists, then $\rho(k) \geq (a+1)10^b - 1$, where $b = [k/9]$ $a = k - 9b$. In the above table, equality holds when $k = 5$, 7, 10, 11, 14, 16, 17, 19, 22, 23, 28, 29, 31, 35, 40, 41, 43, 46, 50, 52, 56, 58, 64, 65, 68, 71 and 79. Does it hold infinitely often? For that matter, is it true that

$$\lim_{k\to\infty} \frac{\rho(k)}{(a+1)10^b - 1} = 1?$$

2. Is it true that $\rho(k) \equiv 9 \pmod{10}$ for all $k > 25$? That $\rho(k) \equiv 99 \pmod{100}$ for all $k > 38$? That $\rho(k) \equiv 999 \pmod{1000}$ for all $k > 59$?

3. The sum of the prime factors of an integer

Let $S(n)$ denote the sum of the prime factors of n taken with multiplicity. A number n is called a *Ruth-Aaron number* if $S(n) = S(n+1)$. It is not known if there exist infinitely many such numbers, a problem which was first studied in 1974 by Nelson, Penney and Pomerance [5].

Similarly, let $\beta(n)$ denote the sum of the distinct prime factors of the integer $n \geq 2$. It is not known if the equation

$$\beta(n) = \beta(n+1) \tag{4}$$

has infinitely many solutions. Moreover, if one could show that there are infinitely many prime numbers p such that the three numbers

$$r = 6p - 1, \quad s = 10p - 1, \quad q = 15p - 4$$

are also prime numbers, then the corresponding number $n = 4pq = rs-1$ would automatically be a solution of $\beta(n) = \beta(n+1)$ $(= 16p-2)$, thus enabling one to construct an infinite family of solutions of (4). The existence of such an infinite sequence of 4-tuples of primes would follow from the *prime k-tuples conjecture* stated by L.E. Dickson [1] in 1904. This conjecture can be written as follows:

If $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ are integers such that $a_i > 0$ and $gcd(a_i, b_i) = 1$, and if for every prime number $p \leq k$, there exists a positive integer n such that none of the integers $a_i n + b_i$ is divisible by p, then there exists an infinite number of positive integers n such that each number $a_i n + b_i$ is prime.

The following table provides the values of $B(x)$, the number of solutions $n \leq x$ of (4), for $x = 10^j$, $j = 1, 2, \ldots, 9$.

x	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
$B(x)$	2	5	10	20	41	140	495	1749	6651

Eventhough no interesting lower bound is known for $B(x)$, upper bounds have been obtained. In 1978, Erdős and Pomerance [2] proved that, if $C(x)$ stands for the number of Ruth-Aaron numbers $\leq x$, then

$$C(x) = O\left(\frac{x}{\log x}\right). \tag{5}$$

Very recently, Pomerance [6] improved this estimate by establishing that

$$C(x) = O\left(\frac{x(\log\log x)^4}{\log^2 x}\right). \tag{6}$$

Now, as Pomerance [6] observed, (5) and (6) also hold if $C(x)$ is replaced by $B(x)$.

Although very little is known about the solutions of (4), it is nevertheless interesting to consider an even more general problem, that is to find, given an arbitrary integer $k \geq 2$, the possible solutions of

$$\beta(n) = \beta(n+1) = \ldots = \beta(n+k-1). \tag{7}$$

In the particular case $k = 3$, the number $n = 89460294$ is a solution of (7) and a computer search reveals that there is no other solution $n < 10^{10}$. Nevertheless, given any integer $k \geq 2$, equations (7) may indeed have infinitely many solutions. In fact, Carl Pomerance (private communication) came up with a heuristic argument indicating that, given any arbitrary small $\varepsilon > 0$, the number $B_k(x)$ of solutions $n \le x$ of (7) satisfies

$$B_k(x) \gg x^{1-\varepsilon}.$$

His argument goes as follows. Let $k \ge 2$ and choose an arbitrary small $\varepsilon > 0$. Let r be a fixed positive integer such that

$$\frac{k-1}{r} < \frac{\varepsilon}{2},$$

and consider the set $\{n \le x : P(n) \le n^{1/r}\}$, which is known to be of density $\rho(r)$, where ρ is the Dickman function (see for instance the book of Tenenbaum [7]). It is reasonable to assume that the set

$$D_k := \{n \le x : \max(P(n), P(n+1), \ldots, P(n+k-1)) \le n^{1/r}\}$$

is also a set of positive density δ_k (with possibly $\delta_k \approx \rho(r)^k$). Since $\beta(n) \le \le P(n)\omega(n) \le P(n)\log n$ for each integer $n \ge 3$, then if $n \in D_k$, we have

$$\beta(n+i) \in [2, n^{1/r}\log(n+k-1)]$$

for $i = 0, 1, \ldots, k-1$. It follows that the probability that (7) holds for a given integer $n \in [3, x]$ is equal to

$$\delta_k \left(n^{1/r}\log(n+k-1)\right)^{-(k-1)} = \delta_k \frac{1}{n^{(k-1)/r}\log^{(k-1)}(n+k-1)} > \delta_k \frac{1}{n^{\varepsilon}},$$

provided n is large enough so that $\log^{(k-1)}(n+k-1) < n^{\varepsilon/2}$. Therefore, if x is sufficiently large, we would have

$$B_k(x) > \delta_k \frac{x}{x^{\varepsilon}} \gg x^{1-\varepsilon}$$

as claimed.

Now consider the function $\beta^*(n) := \beta(n) - P(n)$. It is clearly much smaller than $\beta(n)$ since, denoting by $P_2(n)$ the second largest prime factor of n (or 0 if no such factor exists), we have

$$\beta^*(n) \le (\omega(n)-1)P_2(n) \le (\log n)\sqrt{n}.$$

Hence one might expect the corresponding equations

$$\beta^*(n) = \beta^*(n+1) = \ldots = \beta^*(n+k-1), \qquad (8)$$

to have more solutions. In fact, we can show that (8) has infinitely many solutions n if one accepts the *prime k-tuples conjecture.*

Hence, we shall prove the following result.

Theorem 1. *Let $k \ge 2$ be an integer. If one assumes that the prime k-tuples conjecture is true, then there exist infinitely many positive integers n such that*

$$\beta^*(n-k+1) = \beta^*(n-k+2) = \ldots = \beta^*(n).$$

Proof. Let $k \ge 2$ and $A = \{p \in \mathbf{P} : p \le k\}$. Let $E_1, E_2, \ldots, E_k$ be k sets of primes and R a positive integer satisfying the following conditions:

(a) $(E_i \backslash A) \cap (E_j \backslash A) = \emptyset$ if $i \ne j$;

(b) each prime $q \in A$ belongs to one and only one of the sets $E_i, E_{i+1}, \ldots, E_{i+q-1}$ for each integer $i \in [1, k-q+1]$;

(c) $\sum_{p \in E_1} p = \sum_{p \in E_2} p = \ldots = \sum_{p \in E_k} p = R.$

For the moment, assume that such sets E_i's with an appropriate integer R exist. Then let $Q_i = \prod_{p \in E_i} p$ for $i = 1, 2, \ldots, k$ and consider the system of congruences

$$\begin{cases} n-k+1 \equiv 0 \pmod{Q_1}, \\ n-k+2 \equiv 0 \pmod{Q_2}, \\ \quad\vdots \\ n \equiv 0 \pmod{Q_k}. \end{cases} \qquad (9)$$

Using the Chinese Remainder Theorem, we are guaranteed the existence of a solution $n_0 < Q := Q_1 Q_2 \ldots Q_k$ of (9), each other solution n of (9) being of the form

$$n = n_0 + xQ, \quad \text{with } x = 0, 1, 2, 3, \ldots .$$

Hence, if we set

$$g_i := \gcd\left(\frac{n_0+k-i}{Q_i}, \frac{Q}{Q_i}\right) \quad (1 \le i \le k),$$

then, for each integer $i \in [1, k]$,

$$n-k+i = n_0-k+i+xQ = Q_i g_i \left(\frac{n_0-k+i}{Q_i g_i} + x\frac{Q}{Q_i g_i}\right) = Q_i g_i \cdot q_i(x),$$

where each polynomial $q_i(x)$, $1 \le i \le k$, is defined implicitly. If we can find a value of x, say r, for which the corresponding numbers $q_1(r), q_2(r), \dots, q_k(r)$ are all prime numbers, with

$$q_i(r) > P(Q_i g_i) \quad (1 \le i \le k), \tag{10}$$

then our theorem will be proved, since in this case

$$\beta^*(n-k+i) = \beta^*(Q_i g_i q_i(r)) = \beta(Q_i g_i) = \beta(Q_i) = R \quad (1 \le i \le k).$$

Now

$$\operatorname{pgcd}\left(\frac{n_0-k+i}{Q_i g_i}, \frac{Q}{Q_i g_i}\right) = 1 \quad (1 \le i \le k)$$

and, for each integer $i \in [1,k]$, no prime $p \in A$ divides each value of $q_i(x)$. Hence we may conclude, assuming the *prime k-tuples conjecture*, that there exists one positive integer r (as a matter of fact, infinitely many) such that the corresponding numbers $q_1(r), q_2(r), \dots, q_k(r)$ are all primes, and moreover that they satisfy (10).

Hence, in order to complete the proof of Theorem 1, it remains to show that one can always find k sets of primes $E_1, \dots, E_k$ and in integer R satisfying conditions (a), (b) and (c).

First, observe that if we denote by $s_3(n)$ the number of representations of n as a sum of three primes, then using the known estimate

$$\sum_{q_1+q_2+q_3=n} (\log q_1)(\log q_2)(\log q_3) \sim \frac{n^2}{2} G(n) \quad (n \to \infty),$$

where

$$G(n) = \prod_{p|n}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\nmid n}\left(1 + \frac{1}{(p-1)^2}\right)$$

(see for instance Friedlander and Goldston [3]), one can easily establish that

$$s_3(n) \gg \frac{n^2}{\log^3 n} \quad (n \to \infty).$$

It follows from this estimate that, provided an odd integer n is large enough, the number of representations of n as a sum of three odd primes can be as large as one desires. Using this observation, we now show how one can contruct sets of primes $E_1, E_2, \dots, E_k$ and a corresponding odd integer R satisfying conditions (a), (b) and (c).

Let R be a large odd integer; how large it needs to be will become clear at the end of this argument. Let $F_i = \{r_i, s_i, t_i\}$, $i = 1, 2, \dots, k$, be k sets each containing exactly three distinct primes whose sum is equal to R, and such that $F_i \cap F_j = \emptyset$ if $i \ne j$. Let $R_1, R_2, R_3, \dots$ be odd integers defined implicity by the following relations:

$$\begin{aligned}
2 + R_1 &= R,\\
r_1 + s_1 + t_1 &= R,\\
2+3+5+R_2 &= R,\\
r_2 + s_2 + t_2 &= R,\\
2 + R_1 &= R,\\
3+7+R_3 &= R,\\
2 + R_1 &= R,\\
5+11+R_4 &= R,\\
2+3+13+R_5 &= R,\\
r_3 + s_3 + t_3 &= R,
\end{aligned}$$

and so on, where:

- on each i-th row with i odd, we place the prime 2;
- on each i-th row with $i \ge 3$:
  - if there exists a prime p which has already appeared on the $(i-p)$-th row,
    - we place it on this i-th row;
    - we also place the smallest prime which has not yet appeared on the previous lines, with the additional constraint that we place the two smallest if the left side of the equation is even;
  - if no such prime p exists, we place the next set F_j of primes;
- we end the process when the k-th line has been constructed.

We then define the sets E_i's as follows (where the primes q_i's are primes whose existence is guaranteed by the fact that each odd large integer can be written as the sum of three primes):

$$\begin{aligned}
E_1 &= \{2, q_1, q_2, q_3\}, \quad \text{where } q_1+q_2+q_3 = R_1,\\
E_2 &= \{r_1, s_1, t_1\},
\end{aligned}$$

$$E_3 = \{2,3,5,q_4,q_5,q_6\}, \quad \text{where } q_4+q_5+q_6 = R_2,$$
$$E_4 = \{r_2,s_2,t_2\},$$
$$E_5 = \{2,q_7,q_8,q_9\}, \quad \text{where } q_7+q_8+q_9 = R_3,$$
$$E_6 = \{3,7,q_{10},q_{11},q_{12}\}, \quad \text{where } q_{10}+q_{11}+q_{12} = R_4,$$
$$E_7 = \{2,q_{13},q_{14},q_{15}\}, \quad \text{where } q_{13}+q_{14}+q_{15} = R_5,$$
$$E_8 = \{5,11,q_{16},q_{17},q_{18}\}, \quad \text{where } q_{16}+q_{17}+q_{18} = R_6,$$
$$E_9 = \{2,3,13,q_{19},q_{20},q_{21}\}, \quad \text{where } q_{19}+q_{20}+q_{21} = R_7,$$
$$E_{10} = \{r_3,s_3,t_3\},$$

and so on, up to E_k.

Since R can be taken arbitrarily large, we may assume that each of the primes in the F_i's are larger than the q_i's, which in turn may be assumed to be larger than the p's which we have been adding on some lines. By doing so, we ensure that conditions (a), (b) and (c) are satisfied, thus completing the proof of Theorem 1.

For each positive real number $\varepsilon < 1$, let

$$\beta_\varepsilon(n) := \sum_{\substack{p|n \\ p<n^{1-\varepsilon}}} p.$$

Since, in the above proof, the *prime k-tuples conjecture* guarantees the existence of infinitely many integers x such that $q_i(x)$ is prime for $1 \le i \le k$, and since for each $i \in [1,k]$, the sequence of primes $q_i(x)$ tends to $+\infty$, it is clear that the following result also holds.

Theorem 2. *Given an integer $k \ge 2$ and a positive real number $\varepsilon <$ < 1. Then, assuming the prime k-tuples conjecture, there exist infinitely many positive integers n such that*

$$\beta_\varepsilon(n-k+1) = \beta_\varepsilon(n-k+2) = \ldots = \beta_\varepsilon(n).$$

Using the idea of the proof of Theorem 1, we can also establish that, assuming the *prime k-tuples conjecture*, the quotients $\dfrac{\beta(n-k+j)}{\beta(n-k+i)}$, with $i,j \in$ $\in [1,k]$ can be arbitrarily close to 1, infinitely many often. More precisely, we can prove the following result.

Theorem 3. *Given an integer $k \ge 2$ and assuming the prime k-tuples conjecture,*

$$\liminf_{n\to\infty} \frac{\max\limits_{1\le i\le k} \beta(n-k+i)}{\min\limits_{1\le i\le k} \beta(n-k+i)} = 1. \tag{11}$$

Proof. We only need to show that, given any real number $\delta > 0$, there exist infinitely many integers n such that

$$\frac{\max\limits_{1\le i\le k} \beta(n-k+i)}{\min\limits_{1\le i\le k} \beta(n-k+i)} \le 1+\delta. \tag{12}$$

Let N be a large positive integer and consider the system of congruences

$$\begin{cases} n-k+1 \equiv 0 \pmod{N-k+1}, \\ n-k+2 \equiv 0 \pmod{N-k+2}, \\ \quad\vdots \\ n \equiv 0 \pmod{N}, \end{cases}$$

of which $n_0 = N$ is clearly the smallest positive solution, in which case all other solutions n are given by $n = n_0 + xQ$, $x = 0,1,2,\ldots$, where

$$Q := (N-k+1)(N-k+2)\ldots N.$$

We then have, for each integer $i \in [1,k]$,

$$n-k+i = n_0-k+i+xQ = N-k+i+xQ =$$
$$= (N-k+i)\left(1+x\frac{Q}{N-k+i}\right) = (N-k+i)\cdot q_i(x),$$

say. Assuming the *prime k-tuples conjecture*, we may conclude that there exists a positive integer r such that $q_1(r),\ldots,q_k(r)$ are all prime numbers. For the corresponding integer n, with $\beta(n-k+j) \ge \beta(n-k+i)$, we have

$$1 \le \frac{\beta(n-k+j)}{\beta(n-k+i)} = \frac{\beta((N-k+j)q_j(r))}{\beta((N-k+i)q_i(r))} = \frac{\beta(N-k+j)+q_j(r)}{\beta(N-k+i)+q_i(r)} =$$
$$= \frac{\beta(N-k+j)+(n-k+j)/(N-k+j)}{\beta(N-k+i)+(n-k+i)/(N-k+i)} \le$$
$$\le \frac{\beta(N-k+j)+(n-k+j)/(N-k+j)}{(n-k+i)/(N-k+i)} =$$
$$= \frac{\beta(N-k+j)\dfrac{N-k+j}{n-k+j}+1}{\dfrac{n-k+i}{n-k+j}\cdot\dfrac{N-k+j}{N-k+i}} \le$$
$$\le 1+\delta,$$

provided n is large enough, thus proving Theorem 3.

Computational data. The algorithm mentioned in the proof of Theorem 1 is not optimal; it only guarantees the existence of a set of primes E_i's with an appropriate integer R. Much simpler configurations can be obtained by inspection, say for example for $1 \le k \le 5$. For $k = 2$, one can take $R = 5$ with $E_1 = \{5\}$ and $E_2 = \{2,3\}$ which yields as part of the solutions of (8) the numbers $n = 65$, 185, 365, 785 and 905. For $k = 3$, take $R = 19$ with $E_1 = \{19\}$, $E_2 = \{2,17\}$ and $E_3 = \{3,5,11\}$ providing the solutions

$$n = 8161393,\ 18607213,\ 26068513,\ 64014553 \text{ and } 67212253.$$

For $k = 4$, one can take $R = 36$ with $E_1 = \{2,5,29\}$, $E_2 = \{13,23\}$, $E_3 = \{2,3,31\}$ and $E_4 = \{17,19\}$, with solutions

$$73494447431170, 111699918460090,\ 112793884769890,$$

$$127098796611370 \text{ and } 141955900971130.$$

For $k = 5$, one can take $R = 50$, $E_1 = \{13,37\}$, $E_2 = \{2,5,43\}$, $E_3 = \{3,47\}$, $E_4 = \{2,7,41\}$ and $E_5 = \{19,31\}$, providing the solutions

$$9817058630385 17929,\ 1036978848810729409,\ 1052399282229876529,$$

$$1595959700642651929 \text{ and } 2050448382796747609.$$

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