

On the counting function for the Niven numbers

by

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1. Introduction. A positive integer n is said to be a *Niven number* (or a Harshad number) if it is divisible by the sum of its decimal digits.

In 1984, Kennedy and Cooper [7] established that the set of Niven numbers is of zero density. In 1985, the same authors [1] showed that, given any $t > 0$, we have $N(x) \geq \log^t x$ provided x is sufficiently large, where $N(x)$ stands for the number of Niven numbers not exceeding x , and in 1988, they [2] obtained an asymptotic formula for the number of Niven numbers $\leq x$ whose sum of digits equals k . In 1991, Vardi [9] proved that, for any given $\varepsilon > 0$,

$$N(x) \ll \frac{x}{(\log x)^{1/2-\varepsilon}}$$

and that there exists a positive constant α such that

$$N(x) > \alpha \frac{x}{(\log x)^{11/2}}$$

for infinitely many integers x , namely for all sufficiently large x of the form $x = 10^{10k+n+2}$, k and n being positive integers satisfying $10^n = 45k + 10$.

Recently, De Koninck and Doyon [3] established that, given any fixed $\varepsilon > 0$,

$$x^{1-\varepsilon} \ll N(x) \ll \frac{x \log \log x}{\log x},$$

and conjectured, using a heuristic argument, that, as $x \rightarrow \infty$,

$$(1) \quad N(x) = (\eta + o(1)) \frac{x}{\log x} \quad \text{with} \quad \eta = \frac{14}{27} \log 10.$$

More generally, given an integer $q \geq 2$, we shall say that a positive integer is a *q-Niven number* if it is divisible by the sum of its digits in base q .

2000 *Mathematics Subject Classification*: 11A25, 11A63, 11K65.

The first author was supported by a grant from NSERC.

In this paper, we prove that (1) holds and moreover that, given any base $q \geq 2$, a similar result holds for $N_q(x)$, the number of q -Niven numbers not exceeding x . Hence, our main goal will be to prove the following result.

THEOREM 1. As $x \rightarrow \infty$,

$$(2) \quad N_q(x) = (\eta_q + o(1)) \frac{x}{\log x} \quad \text{with} \quad \eta_q = \frac{2 \log q}{(q-1)^2} \sum_{j=1}^{q-1} (j, q-1).$$

Theorem 1 will follow from our results on the local distribution of $\alpha(n)$, the sum of the digits of n , when n runs over an arithmetic progression with growing modulus k . Similar techniques for the study of the sum of digits function residue classes have been used by other authors, namely Delange [4] and Gel'fond [6].

2. Notations and preliminary observations. Let \mathbb{N} , \mathbb{N}_0 , \mathbb{R} and \mathbb{C} stand for the set of positive integers, non-negative integers, real numbers and complex numbers, respectively.

Throughout this paper, let $q \geq 2$ be a fixed integer. The q -ary expansion of a non-negative integer n is defined as the unique sequence $\epsilon_0(n), \epsilon_1(n), \dots$ for which

$$(3) \quad n = \sum_{j=0}^{\infty} \epsilon_j(n) q^j, \quad \epsilon_j(n) \in \{0, 1, \dots, q-1\}.$$

Let $\alpha(n) = \alpha_q(n)$ be the sum of the digits of n in base q , that is,

$$\alpha(n) = \epsilon_0(n) + \epsilon_1(n) + \dots$$

Given $x \in \mathbb{R}$, $N \in \mathbb{N}$ and $z, w \in \mathbb{C}$, we set

$$(4) \quad S(x|z, w) := \sum_{0 \leq n < x} z^{\alpha(n)} w^n \quad \text{and} \quad S_N(z, w) := S(q^N|z, w).$$

It is clear that

$$(5) \quad S_N(z, w) = \prod_{l=0}^{N-1} \left(\sum_{j=0}^{q-1} z^j w^{jq^l} \right).$$

Let also

$$(6) \quad U(x|z, k, l) := \sum_{\substack{0 \leq n < x \\ n \equiv l \pmod{k}}} z^{\alpha(n)} \quad \text{and} \quad U_N(z, k, l) := U(q^N|z, k, l).$$

Observe that, using the standard notation $e(y) := e^{2\pi iy}$, we have

$$(7) \quad U(x|z, k, l) = \frac{1}{k} \sum_{s=0}^{k-1} e(-ls/k) S(x|z, e(s/k)).$$

Furthermore, if we set

$$(8) \quad A(x|k, l, t) := \#\{n < x : n \equiv l \pmod{k} \text{ and } \alpha(n) = t\},$$

then

$$(9) \quad A(x|k, l, t) = \int_0^1 U(x|e(\xi), k, l) e(-t\xi) d\xi.$$

A function $g : \mathbb{N}_0 \rightarrow \mathbb{C}$ is said to be q -multiplicative if $g(0) = 1$ and

$$g(n) = \prod_{j=0}^{\infty} g(\epsilon_j(n) q^j) \quad (n = 1, 2, \dots).$$

Now for a q -multiplicative function g , set $M(x) = M_q(x) = \sum_{0 \leq n < x} g(n)$. Given a positive integer x , write

$$(10) \quad x = b_1 q^{N_1} + b_2 q^{N_2} + \dots + b_s q^{N_s},$$

where $N_1 > \dots > N_s$, $b_j \in \{1, \dots, q-1\}$. Set

$$\begin{aligned} x_0 &= x, \\ x_1 &= b_2 q^{N_2} + \dots + b_s q^{N_s}, \\ x_2 &= b_3 q^{N_3} + \dots + b_s q^{N_s}, \\ &\vdots \\ x_{s-1} &= b_s q^{N_s}, \\ x_s &= 0 \end{aligned}$$

and

$$\xi_j = \sum_{c=0}^{b_j-1} g(cq^{N_j}) \quad (j = 1, \dots, s).$$

Using these notations, it is easy to observe that

$$(11) \quad M(x) = \xi_1 M(q^{N_1}) + g(b_1 q^{N_1}) M(x_1),$$

and by iteration,

$$(12) \quad M(x) = \xi_1 M(q^{N_1}) + g(b_1 q^{N_1}) \xi_2 M(q^{N_2}) + g(b_1 q^{N_1}) g(b_2 q^{N_2}) \xi_3 M(q^{N_3}) + \dots + g(b_1 q^{N_1}) \dots g(b_{s-1} q^{N_{s-1}}) \xi_s g(b_s q^{N_s}).$$

Note that $S(x|z, w)$ is such a function.

3. Preliminary lemmas. For $y \in \mathbb{R}$, let $\|y\|$ be the distance of y to the closest integer. Let $\xi \in [0, 1)$ be fixed.

LEMMA 1. Let $R \in \mathbb{N}$. Given two coprime positive integers $s < k$ with $(k, q) = 1$ and $k \nmid q-1$, assume that

$$(13) \quad \left\| \xi + \frac{s}{k} q^u \right\| < \frac{1}{8q} \quad \text{for } u = h, h+1, \dots, h+R.$$

Then $q^R \leq k/4$.

Proof. From (13), it follows that

$$(14) \quad \left\| \frac{s}{k} q^u (q-1) \right\| \leq \left\| \left(\xi + \frac{s}{k} q^{u+1} \right) - \left(\xi + \frac{s}{k} q^u \right) \right\| < \frac{1}{4q} \\ (u = h, h+1, \dots, h+R-1).$$

Since $k \nmid q-1$, the left hand side of (14) is non-zero and therefore it is $\geq 1/k$.

Now from (14), we have

$$(15) \quad \left\| \frac{s}{k} q^{u+1} (q-1) \right\| = q \left\| \frac{s}{k} q^u (q-1) \right\| \quad (u = h, h+1, \dots, h+R-2),$$

and therefore

$$(16) \quad \left\| \frac{s}{k} q^{R-1+h} (q-1) \right\| = q^{R-1} \left\| \frac{s}{k} q^h (q-1) \right\| < \frac{1}{4q}.$$

Hence combining this with our observation that the left hand side of (14) must be $\geq 1/k$, we conclude that

$$\frac{1}{k} \leq \left\| \frac{s}{k} q^h (q-1) \right\| < \frac{1}{4q^R},$$

that is $q^R \leq k/4$, as claimed.

LEMMA 2. Let $A(x|k, l, t)$ be as in (8) and $S(x|z, w)$ as in (4). Then

$$\left| A(x|k, l, t) - \frac{1}{k} A(x|1, 0, t) \right| \leq \max_{1 \leq s \leq k-1} \max_{|z|=1} |S(x|z, e(s/k))|.$$

Proof. This follows immediately from (9) and (7).

Now for $1 \leq s < k$, set

$$s_h = \max_{0 \leq j \leq q-1} \left\| j\xi + q^h \frac{s}{k} \right\|.$$

LEMMA 3. There exists a constant $c = c(q)$ such that

$$\left| \frac{1}{q} \sum_{j=0}^{q-1} e(\xi j) e\left(\frac{s}{k} q^h j\right) \right| \leq q^{-cs_h}.$$

Proof. This follows immediately from the definition of s_h .

4. Local distribution of $\alpha(n)$ as n runs through a congruence class $l \pmod{k}$

4.1. We first consider the case $(k, q(q-1)) = 1$.

THEOREM 2. Assume that $(k, q(q-1)) = 1$. Then, for each integer $l \in [0, k-1]$ and $t \in \mathbb{N}$, we have

$$(17) \quad \left| A(x|k, l, t) - \frac{1}{k} A(x|1, 0, t) \right| \leq x e^{-c_1 \frac{\log x}{\log 2k}},$$

where $c_1 = c_1(c, q)$ is a suitable positive constant independent of k, l and t .

Proof. Let x be written as in (10). Then, from (12), we have

$$|S(x|z, e(s/k))| \leq q \sum_{j=1}^s |S_{N_j}(z, e(s/k))|.$$

To estimate each expression $|S_{N_j}(z, e(s/k))|$, we use Lemmas 1–3.

For $k = 2, 3, 4$, we set $R = 0$, while for each $k \geq 5$, we set

$$R = \left\lceil \frac{\log(kq/4)}{\log q} \right\rceil.$$

From Lemma 1, we know that

$$\max_{h \leq u \leq h+R} s_u \geq \frac{1}{8q}.$$

Therefore

$$|S_{N_j}(z, e(s/k))| \leq q^{N_j} \cdot q^{-\frac{c}{8q} \left\lceil \frac{N_j}{R+1} \right\rceil},$$

which completes the proof of Theorem 2.

REMARK. It is interesting to observe that the following assertion is also true:

If $(k, q(q-1)) = 1$, then

$$\max_{|z|=1} \left| \sum_{\substack{n < x \\ n \equiv l \pmod{k}}} z^{\alpha(n)} - \frac{1}{k} \sum_{n < x} z^{\alpha(n)} \right| \leq x e^{-c_1 \frac{\log x}{\log 2k}}.$$

4.2. We now consider the case $(k, q) > 1$. Actually we shall reduce this case to the one of Section 4.1. Indeed, let $k = k_1 k_2$, where k_1 is the largest divisor of k coprime to q and $k_2 = k/k_1$. Further let h be the smallest positive integer such that $k_2 | q^h$. Then the congruence class $l \pmod{k}$ can be written as the union of some congruence classes $\pmod{k_1 q^h}$, namely

$$(18) \quad \{n : n \equiv l \pmod{k}\} = \bigcup_{j=1}^{q^h/k_2} \{n : n \equiv l^{(j)} \pmod{k_1 q^h}\}.$$

First define $l_1^{(j)}$ and $l_2^{(j)}$ implicitly by

$$l^{(j)} = l_1^{(j)} + q^h l_2^{(j)}, \quad 0 \leq l_1^{(j)} < q^h,$$

and then write a positive integer $n \equiv l^{(j)} \pmod{k_1 q^h}$ as

$$n = l_1^{(j)} + q^h m \equiv l_1^{(j)} + q^h l_2^{(j)} \pmod{k_1 q^h},$$

which is equivalent to

$$(19) \quad m \equiv l_2^{(j)} \pmod{k_1}.$$

Using this setup, we obtain the following result.

LEMMA 4. We have

$$(20) \quad \sum_{\substack{n < x \\ n \equiv l \pmod{k}}} z^{\alpha(n)} = \sum_{j=1}^{q^h/k_2} z^{\alpha(l_1^{(j)})} \sum_{\substack{m < x/q^h \\ m \equiv l_2^{(j)} \pmod{k_1}}} z^{\alpha(m)}$$

and

$$(21) \quad A(x|k, l, t) = \sum_{j=1}^{q^h/k_2} A\left(\frac{x}{q^h} \Big| k_1, l_2^{(j)}, t - \alpha(l_1^{(j)})\right).$$

4.3. We now consider the case $k = k_1 k_2$, where $(k, q) = 1$, $(k_1, q-1) = 1$ and all the prime factors of k_2 are divisors of $q-1$.

LEMMA 5. We have

$$(22) \quad U(x|z, k, l) = \frac{1}{k} \sum_{\tau=1}^{k_2} e(-l\tau/k_2) S(x|z, e(\tau/k_2)) + O(xe^{-c_1 \frac{\log x}{\log 2k}})$$

and

$$(23) \quad U(x|z, k, l) = \frac{1}{k_1} U(x|z, k_2, l) + O(xe^{-c_1 \frac{\log x}{\log 2k}}).$$

Proof. It is clear that (23) follows from (22) and (7). Therefore we only need to prove (22). Recall the representation of $U(x|z, k, l)$ given by (7). For each $1 \leq s < k$, write $s/k = s^*/k^*$, where $(s^*, k^*) = 1$. If k^* has a prime factor which does not divide k_2 , then arguing as in the proof of Theorem 2, we obtain

$$|S(x|z, e(s/k))| \leq xe^{-c_1 \frac{\log x}{\log 2k}}.$$

Therefore, it remains only to consider those s which are multiples of k_1 , in which case we simply write $s = \tau k_1$, where $\tau = 0, 1, \dots, k_2 - 1$, and the proof is complete.

COROLLARY. If $k = k_1 k_2$ with $(k, q) = 1$, $(k_1, q-1) = 1$ and all the prime factors of k_2 are divisors of $q-1$, then

$$(24) \quad A(x|k, l, t) = \frac{1}{k_1} A(x|k_2, l, t) + O(xe^{-c_1 \frac{\log x}{\log 2k}}).$$

4.4. Assume now that the prime divisors of k divide $q-1$. For each positive integer m , let $\kappa(m) = (m, q-1)$ and set $K = k/\kappa(k)$. Then, repeating

the argument used above and again using Lemmas 1–3, we can conclude that

$$A(x|k, l, t) = \frac{1}{K} A(x|\kappa(k), l, t) + O(xe^{-c_1 \frac{\log x}{\log 2k}}).$$

4.5. Assume finally that $k|q-1$. Since in this case, we have $q^\nu \equiv 1 \pmod{k}$ for each $\nu \in \mathbb{N}_0$, it follows that $n \equiv l \pmod{k}$ implies that $\alpha(n) \equiv l \pmod{k}$. Consequently,

$$(25) \quad A(x|k, l, t) = \begin{cases} \#\{n < x : \alpha(n) = t\} & \text{if } t \equiv l \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

We now have the proper setup to build the proof of Theorem 1.

5. The proof of Theorem 1. Given x , define N_x as the unique integer satisfying $q^{N_x} \leq x < q^{N_x+1}$, so that $N_x = \left\lfloor \frac{\log x}{\log q} \right\rfloor$.

Further define

$$B(x|t) := \#\{n < x : \alpha(n) = t \text{ with } t|n\},$$

$$a(x|t) := A(x|1, 0, t) = \#\{n < x : \alpha(n) = t\}.$$

Using Theorem 6, Chapter VII, of V. V. Petrov [8] on local distribution of sums of identically distributed random variables, and by an easy computation we obtain the following.

LEMMA 6. Let

$$m = \frac{q-1}{2} \quad \text{and} \quad \sigma^2 = \frac{1}{q} \sum_{j=1}^{q-1} j^2 - m^2 = \frac{q^2-1}{12}.$$

Then

$$(26) \quad a(x|t) = \frac{x}{\sqrt{N_x}} \varphi\left(\frac{t - mN_x}{\sigma\sqrt{N_x}}\right) + O\left(\frac{x(\log N_x)^{3/2}}{N_x}\right)$$

uniformly in t , where $\varphi(y) = (1/\sqrt{2\pi})e^{-y^2/2}$ is the density function of the Gaussian law.

REMARK. For a similar result in a more general setup, see Drmota and Gajdosik [5].

Now, x being fixed, we define the interval I as follows:

$$I = \left[\frac{q-1}{2} N_x - \frac{N_x}{\log^2 N_x}, \frac{q-1}{2} N_x + \frac{N_x}{\log^2 N_x} \right].$$

A simple probabilistic argument shows that

$$(27) \quad \#\{n < x : \alpha(n) \notin I\} \ll \frac{x}{\log x \log \log x}.$$

Therefore, it is clear that

$$(28) \quad N_q(x) = \sum_{t \in I} B(x|t) + O\left(\frac{x}{\log x \log \log x}\right).$$

Let us factorise each $t \in I$ as $t = t_1 t_2 t_3$, where $(t_1, q(q-1)) = 1$, the prime factors of t_2 divide q , and the prime factors of t_3 divide $q-1$.

Fixing $t \in I$, let h be the smallest positive integer such that $t_2 | q^h$. Note that

$$(29) \quad q^h < N_x^{c_3} \quad \text{for a suitable positive constant } c_3 = c_3(q).$$

To see this, first observe that t_2 must have a divisor to the h -th power, and therefore $N_x > t_2 \geq 2^h$, which means that $h < \log N_x / \log 2$. Hence $q^h < q^{\log N_x / \log 2} < N_x^{c_3}$, which proves (29).

Using (21), we obtain

$$(30) \quad A(x|t, 0, t) = \sum_{j=1}^{q^h/t_2} A\left(\frac{x}{q^h} \middle| t_1 t_3, l_2^{(j)}, t - \alpha(l_1^{(j)})\right),$$

where

$$(31) \quad l^{(j)} := (t_1 t_3) t_2 j = l_1^{(j)} + q^h l_2^{(j)} \quad (0 \leq l_1^{(j)} < q^h).$$

Using (24), we have

$$(32) \quad A\left(\frac{x}{q^h} \middle| t_1 t_3, l_2^{(j)}, t - \alpha(l_1^{(j)})\right) \\ = \frac{1}{t_1} A\left(\frac{x}{q^h} \middle| t_3, l_2^{(j)}, t - \alpha(l_1^{(j)})\right) + O\left(\frac{x}{q^h} e^{-\frac{c_1}{2} \cdot \frac{\log x}{\log 2t}}\right).$$

Since $\kappa(t_3)$ divides t and $l^{(j)}$, $\alpha(l_1^{(j)}) \equiv l_1^{(j)} \pmod{\kappa(t_3)}$, $l^{(j)} = l_1^{(j)} + q^h l_2^{(j)}$ and $q^h \equiv 1 \pmod{\kappa(t_3)}$, it follows that

$$t \equiv \alpha(l_1^{(j)}) \equiv l_2^{(j)} \pmod{\kappa(t_3)}.$$

Therefore the main term on the right hand side of (32) is, because of (25),

$$\frac{1}{t_1} \cdot \frac{\kappa(t_3)}{t_3} a\left(\frac{x}{q^h} \middle| t - \alpha(l_1^{(j)})\right).$$

Consequently, using (30), we obtain

$$(33) \quad A(x|t, 0, t) = \frac{\kappa(t_3)}{t_1 t_3} \sum_{j=1}^{q^h/t_2} a\left(\frac{x}{q^h} \middle| t - \alpha(l_1^{(j)})\right) + O\left(x e^{-\frac{c_1}{2} \cdot \frac{\log x}{\log 2t}}\right).$$

Using Lemma 6, and after observing that

$$(34) \quad \begin{aligned} l_1^{(j)} &< q^h < N_x^{c_3}, \\ \alpha(l_1^{(j)}) &= O(\log l_1^{(j)}) = O(\log N_x), \\ |\varphi(\xi_1) - \varphi(\xi_2)| &\ll |\xi_1 - \xi_2|, \end{aligned}$$

we find that, for each $t \in I$,

$$(35) \quad a\left(\frac{x}{q^h} \middle| t - \alpha(l_1^{(j)})\right) = a\left(\frac{x}{q^h} \middle| t\right) + O\left(\frac{x}{q^h} \cdot \frac{(\log N_x)^{3/2}}{N_x}\right).$$

Therefore, using (33),

$$(36) \quad A(x|t, 0, t) = \frac{q^h \kappa(t_3)}{t} a\left(\frac{x}{q^h} \middle| t\right) + O\left(\frac{x}{t} \cdot \frac{(\log N_x)^{3/2}}{N_x}\right).$$

Furthermore, by Lemma 6, we have

$$(37) \quad \left| q^h a\left(\frac{x}{q^h} \middle| t\right) - a(x|t) \right| \\ \ll \left| \frac{x}{\sqrt{N_x - h}} \varphi\left(\frac{t - m(N_x - h)}{\sigma \sqrt{N_x - h}}\right) - \frac{x}{\sqrt{N_x}} \varphi\left(\frac{t - mN_x}{\sigma \sqrt{N_x}}\right) \right| \\ + O\left(\frac{x}{N_x} (\log N_x)^{3/2}\right).$$

But the expression $|\dots|$ on the right hand side of (37) is no larger than the error term, which implies that

$$(38) \quad \left| q^h a\left(\frac{x}{q^h} \middle| t\right) - a(x|t) \right| \ll \frac{x}{N_x} (\log N_x)^{3/2}.$$

Hence, using (36) and (38), we obtain

$$(39) \quad A(x|t, 0, t) = \frac{\kappa(t_3)}{t} a(x|t) + O\left(\frac{x}{t N_x} (\log N_x)^{3/2}\right).$$

From (28) and (39), we then have, since $N_x = \lfloor \log x / \log q \rfloor$,

$$(40) \quad \begin{aligned} N_q(x) &= \sum_{t \in I} \frac{\kappa(t_3)}{t} a(x|t) + O\left(\frac{x}{N_x \log^2 N_x} (\log N_x)^{3/2}\right) \\ &= \frac{2}{N_x(q-1)} \sum_{t \in I} \kappa(t_3) a(x|t) + O\left(\frac{x}{(\log x)(\log \log x)^{1/2}}\right) \\ &= \frac{2 \log q}{\log x} \cdot \frac{1}{q-1} \sum_{t \in I} \kappa(t_3) a(x|t) + O\left(\frac{x}{(\log x)(\log \log x)^{1/2}}\right). \end{aligned}$$

Since $a(x|t) = (1 + o(1))a(x|t+1)$ uniformly for $t \in I$, $\kappa(t_3) = \kappa(t)$, and

$\kappa(t)$ is periodic mod $q-1$, it follows that

$$(41) \quad \sum_{t \in I} \kappa(t_3) a(x|t) = \frac{1}{q-1} (1 + o(1)) \sum_{t \in I} \kappa(t) \sum_{j=0}^{q-2} a(x|t-j) \\ = (1 + o(1)) \sum_{r \in I} a(x|r) \cdot \frac{1}{q-1} \sum_{j=0}^{q-2} \kappa(r+j) + E(x),$$

where $E(x) \ll \sum' a(x|s)$, where this last sum runs over those s such that $|s - I_i| \leq q-1$, the I_i 's being the endpoints of I , that is, $I = [I_1, I_2]$. Since $\max a(x|t) \ll x/\sqrt{\log x}$ and since the number of s 's counted in $\sum' a(x|s)$ is bounded by a multiple of q , it follows that

$$(42) \quad E(x) \ll \frac{x}{\sqrt{\log x}}.$$

Moreover, observe that, because of (27),

$$(43) \quad \sum_{r \in I} a(x|r) = x + O\left(\frac{x}{\log x \log \log x}\right).$$

Finally, observe that

$$(44) \quad \frac{1}{q-1} \sum_{j=0}^{q-2} \kappa(r+j) = \frac{1}{q-1} \sum_{j=1}^{q-1} \kappa(j)$$

is a constant.

Therefore, it follows from (40)–(44) that

$$N_q(x) = (1 + o(1)) \frac{2x}{\log x} \cdot \frac{1}{(q-1)^2} \sum_{j=1}^{q-1} \kappa(j),$$

which implies (2). The proof of Theorem 1 is thus complete.

6. Final remark. A similar result can be established if one replaces $\alpha(n)$ by a q -additive function $f(n)$ taking integer values and satisfying $f(bq^j) = f(b)$ for all positive integers j .

Acknowledgements. The authors would like to thank the referee whose remarks helped improve the quality of this paper.

References

- [1] C. N. Cooper and R. E. Kennedy, *On an asymptotic formula for the Niven numbers*, *Internat. J. Math. Math. Sci.* 8 (1985), 537–543.
 [2] —, —, *A partial asymptotic formula for the Niven numbers*, *Fibonacci Quart.* 26 (1988), 163–168.

- [3] J. M. De Koninck and N. Doyon, *On the number of Niven numbers up to x* , *ibid.*, to appear.
 [4] H. Delange, *Sur les fonctions q -additives ou q -multiplicatives*, *Acta Arith.* 21 (1972), 285–298.
 [5] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, *J. Théor. Nombres Bordeaux* 10 (1998), 17–32.
 [6] A. O. Gel'fond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, *Acta Arith.* 13 (1967/1968), 259–265.
 [7] R. E. Kennedy and C. N. Cooper, *On the natural density of the Niven numbers*, *College Math. J.* 15 (1984), 309–312.
 [8] V. V. Petrov, *Sums of Independent Random Variables*, Springer, 1975.
 [9] I. Vardi, *Computational Recreations in Mathematics*, Addison-Wesley, Redwood City, CA, 1991.

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Received on 20.11.2001
 and in revised form on 18.2.2002

(4150)