Arithmetic functions defined on sets of primes of positive density

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Abstract

Several asymptotic formulas are proved for arithmetic sums, which involve the largest prime factor of an integer and certain large additive functions. All the functions are defined on a set of primes having density δ $(0<\delta<1)$ in the set of all primes.

1 Introduction

Let Q be a set of primes such that there exists some constant δ satisfying $0<\delta<1$ and

$$\pi(x,Q) := \sum_{p\le x,\ p\in Q} 1 = \delta \mathrm{Li}\, x + O\left(\frac{x}{\log^B x}\right). \tag{1.1}$$

Here and later p denotes primes, $\mathrm{Li}\, x = \int_2^x \frac{dt}{\log t}$, and B is a constant satisfying $B>2$. It is possible to treat the case when one assumes only $B>1$ in (1.1) (see R. Warlimont [14]), but as in [3] and [11] we find it sufficient to assume $B>2$ in (1.1). In fact, the present work may be considered as a continuation of the first author's work [3] and the second author's [11]. All the relevant notation from these papers will be retained here. We define $P(n,Q)$ as

$$P(n,Q) = \begin{cases} \max\{p : p|n \wedge p\in Q\} & \text{if } (n,Q)>1, \\ 0 & \text{otherwise,}\end{cases} \tag{1.2}$$

where $(n,Q)>1$ (resp. $(n,Q)=1$) means that n has a prime factor (resp. has no prime factor) from Q. Thus $P(n,Q)$ is the largest prime factor of n belonging to Q, and analogously we define the k-th largest prime factor of n belonging to Q as

$$P_k(n,Q) = \begin{cases} P\left(\frac{n}{P_1(n,Q)\ldots P_{k-1}(n,Q)}, Q\right) & \text{if } \Omega(n,Q)\ge k, \\ 0 & \text{otherwise,}\end{cases} \tag{1.3}$$

if $k\ge 2$, where $P_1(n,Q)\equiv P(n,Q)$ and

$$\Omega(n,Q) = \sum_{p^\alpha \| n,\ p\in Q} \alpha \tag{1.4}$$

is the total number of prime factors of n belonging to Q, while

$$\omega(n,Q) = \sum_{p|n,\ p\in Q} 1 \tag{1.5}$$

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is the number of distinct prime factors of n belonging to Q. Here as usual $p^\alpha \| n$ means that p^α divides n, but $p^{\alpha+1}$ does not. The functions defined by (1.2)-(1.5) are the analogues of the classical functions

$$P(n) = \max\{p : p|n\}, \quad P_k(n) = \begin{cases} P\left(\frac{n}{P_1(n)\ldots P_{k-1}(n)}\right) & \text{if } \Omega(n) \geq k, \\ 0 & \text{otherwise,} \end{cases}$$

if $k \geq 2$, and

$$\Omega(n) = \sum_{p^\alpha \| n} \alpha p, \quad \omega(n) = \sum_{p|n} 1.$$

Likewise in [11] we defined large additive functions

$$\beta(n, Q) = \sum_{p|n,\ p \in Q} p, \quad B(n, Q) = \sum_{p^\alpha \| n,\ p \in Q} \alpha p, \tag{1.6}$$

and $\beta(n, Q) = B(n, Q) = 0$ if $(n, Q) = 1$. The functions in (1.6) are the analogues of the large additive functions

$$\beta(n) = \sum_{p|n} p, \quad B(n) = \sum_{p^\alpha \| n} \alpha p,$$

for which there exists an extensive literature (e.g. see the monograph [4] and the papers [1], [5], [7], [8], [12], [15], where references to other works may be found).

In [3] the first author proved

$$\sum_{n \leq x}{}' \frac{1}{P(n, Q)} = \left(\eta(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta}, \tag{1.7}$$

where $\eta(Q)$ is a positive constant depending on Q (i.e. δ) which may be written down in closed form. In general $\sum'_{n \leq x} 1/f(n)$ denotes the sum over n not exceeding x for which $f(n) \neq 0$, so that

$$\sum_{n \leq x}{}' \frac{1}{P(n, Q)} = \sum_{n \leq x,\ (n,Q)>1} \frac{1}{P(n, Q)}.$$

Several results involving $\beta(n, Q)$ and $B(n, Q)$ were established by the second author [11]. For instance, it was proved that

$$\sum_{n \leq x} \beta(n, Q) = \sum_{j \leq B} \frac{\delta A_j x^2}{\log^j x} + O\left(\frac{x^2}{\log^B x}\right) \tag{1.8}$$

with explicitly given constants A_j,

$$\sum_{n \leq x} (B(n, Q) - \beta(n, Q)) = \delta x \log\log x + E(Q) + x \sum_{j \leq B} \frac{E_j(Q)}{\log^j x} + O\left(\frac{x}{\log^B x}\right), \tag{1.9}$$

$$\sum_{n \leq x}{}' \frac{1}{B(n, Q) - \beta(n, Q)} = A(Q)x + O\left(x^{\frac{1}{2}} \log x\right) \quad (A(Q) > 0), \tag{1.10}$$

and

$$\sum_{n\le x}{}' \frac{B(n,Q)}{\beta(n,Q)} = x + O\left(\frac{x\log\log x}{(\log x)^{\delta}}\right), \tag{1.11}$$

where the constants in (1.9) and (1.10) are effectively computable, and in view of (1.9) it is seen that (1.8) remains valid if $\beta(n,Q)$ is replaced by $B(n,Q)$. Moreover it was conjectured in [11] that

$$\sum_{n\le x}{}' \frac{1}{\beta(n,Q)} = \left(\eta_1(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}, \tag{1.12}$$

and

$$\sum_{n\le x}{}' \frac{1}{B(n,Q)} = \left(\eta_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}, \tag{1.13}$$

with $0 < \eta_2(Q) \le \eta_1(Q) \le \eta(Q)$, where $\eta(Q)$ is the constant appearing in (1.7). It is the aim of this paper to establish the asymptotic formulas (1.12) and (1.13), and to prove some other results involving the functions $\beta(n,Q)$, $B(n,Q)$ and $P_k(n,Q)$.

2 Statement of results

K. Alladi and P. Erdős [2] proved that, for any fixed $k \ge 2$,

$$\sum_{2\le n\le x} \frac{P_k(n)}{P(n)} = (1+o(1))a_k\frac{x}{(\log x)^{k-1}} \qquad (x\to\infty), \tag{2.1}$$

where the a_k's are effectively computable positive constants. Thus the asymptotic behaviour of the sum in (2.1) changes with k. However all the sums of $P_k(n,Q)/P(n,Q)$ are of the same order of magnitude, which shows a completely different behaviour. Our result is contained in

Theorem 1. *For any fixed integer $k \ge 2$ we have, with a suitable constant $C_k(\delta) > 0$,*

$$\sum_{n\le x,\ (n,Q)>1} \frac{P_k(n,Q)}{P(n,Q)} = \left(C_k(\delta) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.2}$$

Actually it will transpire from the proof that

$$C_k(\delta) = C(Q)\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\substack{p_1\ge P(m)\\ p_1\in Q}}\ell_Q(p_1)\sum_{\substack{p_2\ge p_1\\ p_2\in Q}}\frac{1}{p_2}\cdots\sum_{\substack{p_{k-1}\ge p_{k-2}\\ p_{k-1}\in Q}}\frac{1}{p_{k-1}}\sum_{\substack{p_k\ge p_{k-1}\\ p_k\in Q}}\frac{1}{p_k^2}, \tag{2.3}$$

where $C(Q)$ is the constant appearing in Lemma 4, $\ell_Q(y)$ is given by (3.8), and p_1, p_2, $\ldots$, p_k denote primes. The multiple series in (2.3) is easily seen to be convergent by the prime number theorem, Lemma 6 and Lemma 2. The method of proof of Theorem 1 may be used to treat some other arithmetic sums, such as

$$\sum_{n\le x,\ \Omega(n,Q)\ge m} \frac{P_k(n)}{P_m(n)}, \tag{2.4}$$

where $k > m \ge 1$ are fixed integers. Also this method may be used to treat two sums related to the sum in (1.11). We shall prove

Theorem 2. *There exist constants* $0 < D_1(\delta) < D_2(\delta)$ *such that*

$$\sum_{n\le x,\ (n,Q)>1} \frac{\beta(n,Q)}{P(n,Q)} = x + \left(D_1(\delta) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.5}$$

and

$$\sum_{n\le x,\ (n,Q)>1} \frac{B(n,Q)}{P(n,Q)} = x + \left(D_2(\delta) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.6}$$

The explicit expressions for $D_1(\delta)$ and $D_2(\delta)$ will be given in the proof. The next result establishes the asymptotic formulas (1.12) and (1.13). This is

Theorem 3. *There exist constants* $0 < \eta_2(Q) < \eta_1(Q)$ *such that*

$$\sum_{n\le x,\ (n,Q)>1} \frac{1}{\beta(n,Q)} = \left(\eta_1(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}} \tag{2.7}$$

and

$$\sum_{n\le x,\ (n,Q)>1} \frac{1}{B(n,Q)} = \left(\eta_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^{\delta}}. \tag{2.8}$$

The asymptotic formulas (2.7) and (2.8) display the difference in behaviour of $\beta(n)$ (resp. $B(n)$) and $\beta(n,Q)$ (resp. $B(n,Q)$), since it is known that

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} = x\exp\left\{-(2\log x\log\log x)^{1/2} + O\left((\log x\log\log\log x)^{1/2}\right)\right\}. \tag{2.9}$$

The asymptotic formula (2.9), which remains true if $\beta(n)$ is replaced by $B(n)$ or $P(n)$, was proved in [10], and then sharpened in [12] and [8]. This is analogous to the difference in behaviour between the sum of reciprocals of $P(n)$ and $P(n,Q)$, as noted in [3] and [11]. The difference in behaviour between $P(n)$ and $P(n,Q)$ is also reflected in the asymptotic behaviour of two further arithmetic sums which contain the logarithms of these functions. The results are

Theorem 4. *There exists an effectively computable constant B such that*

$$\sum_{2\le n\le x}\frac{1}{n\log P(n)} = e^{\gamma}\log\log x + B + O\left(\frac{1}{\log x}\right), \tag{2.10}$$

where γ is Euler's constant.

Theorem 5. *There exists an effectively computable constant $F(Q) > 0$ such that*

$$\sum_{n\le x,\ (n,Q)>1}\frac{1}{n\log P(n,Q)} = \left(F(Q) + O\left(\frac{1}{\log\log x}\right)\right)\log^{1-\delta} x. \tag{2.11}$$

Theorem 4 sharpens a result of De Koninck - Sitaramachandrarao [6], who obtained

$$\sum_{2\le n\le x}\frac{1}{n\log P(n)} = e^{\gamma}\log\log x + O(1);$$

their paper contains a discussion on earlier results on this problem. Perhaps the bound for the error term in (2.10) is of the correct order of magnitude.

3 The necessary lemmas

This section is devoted to the lemmas needed in the sequel.

Lemma 1. *For $2 \le y \le x$ we have uniformly*

$$\psi(x,y) = \sum_{n\le x,\ P(n)\le y} 1 \ll x\exp\left(-\frac{\log x}{2\log y}\right), \tag{3.1}$$

while for $\exp\left((\log\log x)^{5/3+\varepsilon}\right) \le y \le x$ we have the asymptotic formula

$$\psi(x,y) = x\rho(u)\left\{1 + O\left(\frac{\log(u+2)}{\log y}\right)\right\}, \quad u = \frac{\log x}{\log y}, \tag{3.2}$$

where the error term is uniform. The Dickman - de Bruijn function $\rho(u)$ is the continuous solution of the equation $u\rho'(u) = -\rho(u-1)$ with the initial condition $\rho(u) = 1$ for $0 \le u \le 1$. It satisfies

$$\rho(u) = \exp\{-u(\log u + \log\log u + O(1))\}. \tag{3.3}$$

These are standard results on $\psi(x,y)$, to be found e.g. in G. Tenenbaum [13].

Lemma 2. *For $\xi > 1$ fixed,*

$$\sum_{n>x} \frac{1}{n(\log P(n))^\xi} \ll_\xi \frac{1}{\log^{\xi-1} x}.$$

Proof. By partial summation the above sum may be written as

$$\begin{aligned}
\sum_p \frac{1}{p\log^\xi p} \sum_{\substack{m>\frac{x}{p}\\ P(m)\le p}} \frac{1}{m} &= \sum_p \frac{1}{p\log^\xi p}\left(\frac{\psi(\frac{x}{p},p)}{\frac{x}{p}} + \int_{x/p}^\infty \frac{\psi(t,p)}{t^2}\,dt\right)\\
&= \Sigma_1 + \Sigma_2,
\end{aligned}$$

where in Σ_1 summation is over $p \le \sqrt{x}$, and in Σ_2 over $p > \sqrt{x}$. Then using (3.1) we obtain, after change of variable $\frac{\log t}{\log p} = u$, $\frac{\log x}{\log v} = y$,

$$\begin{aligned}
\Sigma_1 &\ll \sum_{p\le\sqrt{x}} \frac{1}{p\log^\xi p}\left(e^{-\frac{\log x}{2\log p}} + \int_{x/p}^\infty e^{-\frac{\log t}{2\log p}}\frac{dt}{t}\right)\\
&\ll \sum_{p\le\sqrt{x}} \frac{1}{p\log^{\xi-1} p} e^{-\frac{\log x}{2\log p}} = \int_2^{\sqrt{x}} \frac{1}{v\log^\xi v} e^{-\frac{\log x}{2\log v}}\,dv + O_\xi\left(\frac{1}{\log^{\xi-1} x}\right)\\
&= \frac{1}{\log^{\xi-1} x}\left(\int_2^{\log x/\log 2} y^{\xi-2} e^{-y/2}\,dy + O_\xi(1)\right) \ll_\xi \frac{1}{\log^{\xi-1} x},
\end{aligned}$$

where we used the prime number theorem in the form

$$\pi(x) = \sum_{p\le x} 1 = \operatorname{Li} x + \Delta(x), \quad \Delta(x) = O\left(xe^{-\sqrt{\log x}}\right). \tag{3.4}$$

The trivial estimate $\psi(x,y) \le x$ gives

$$\begin{aligned}
\Sigma_2 &= \sum_{p>\sqrt{x}} \frac{1}{p\log^\xi p}\left(\frac{\psi(\frac{x}{p},p)}{\frac{x}{p}} + \int_{x/p}^\infty \frac{\psi(t,p)}{t^2}\,dt\right)\\
&\ll_\xi \frac{1}{\log^\xi x} + \sum_{p>\sqrt{x}} \frac{1}{p\log^\xi p}\left(\int_{x/p}^p \frac{dt}{t} + \int_p^\infty \frac{\psi(t,p)}{t^2}\,dt\right)\\
&\ll_\xi \frac{1}{\log^{\xi-1} x} + \sum_{p>\sqrt{x}} \frac{1}{p\log^\xi p}\int_p^\infty \frac{\psi(t,p)}{t^2}\,dt.
\end{aligned}$$

Finally by Lemma 1 the last sum is

$$\ll \sum_{p>\sqrt{x}} \frac{1}{p\log^\xi p}\int_p^\infty e^{-\frac{\log t}{2\log p}}\frac{dt}{t} \ll \sum_{p>\sqrt{x}} \frac{1}{p\log^{\xi-1} p} \ll_\xi \frac{1}{\log^{\xi-1} x}.$$

Remark. By elaborating the method of proof given above, it can be shown that there exists $\kappa(\xi) > 0$ such that

$$\sum_{n>x} \frac{1}{n(\log P(n))^\xi} = \frac{\kappa(\xi)}{\log^{\xi-1} x} + O\left(\frac{(\log\log x)^{2\xi}}{\log^\xi x}\right).$$

Lemma 3. *If $p(n)$ is the smallest prime factor of an integer $n \geq 2$, then uniformly for $2 \leq y \leq x$*

$$\sum_{n\leq x,\ p(n)>y} 1 \ll \frac{x}{\log y}. \tag{3.5}$$

This is a well-known sieve bound. For a thorough discussion of estimates for the sum in (3.5) the reader is referred to G. Tenenbaum [13].

Lemma 4. *There is a positive constant $C(Q)$ such that*

$$\sum_{n\leq x,\ (n,Q)=1} 1 = \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}. \tag{3.6}$$

This is Lemma 5 of De Koninck [3], and follows from the work of Goldston - McCurley [9].

Lemma 5. *If $2 \leq y \leq e^{\log^\alpha x}$ for some $0 < \alpha < 1$, then uniformly*

$$\sum_{n\leq x,\ (n,Q)>1,\ p(n)>y} 1 = \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right)\ell_Q(y)\frac{x}{(\log x)^\delta}, \tag{3.7}$$

where $C(Q)$ is the constant appearing in (3.6), and

$$\ell_Q(y) = \prod_{p\leq y,\ p\notin Q}\left(1 - \frac{1}{p}\right). \tag{3.8}$$

This follows, in the special case when $y \leq e^{\log^\alpha x}$, from Lemma 7 of [3].

Lemma 6. *If $\ell_Q(y)$ is given by (3.8), then there is a positive constant ν_Q such that*

$$\ell_Q(y) = \left(\nu_Q + O\left(\frac{1}{\log y}\right)\right)\log^{\delta-1} y. \tag{3.9}$$

This is Lemma 8(i) of [3]. One has

$$\log \ell_Q(y) = \sum_{p\le y,\ p\notin Q} \log\left(1-\frac{1}{p}\right) \tag{3.10}$$
$$= -\sum_{p\le y,\ p\notin Q} \frac{1}{p} - \sum_{m\ge 2}\frac{1}{m}\sum_{p\le y,\ p\notin Q}\frac{1}{p^m}$$
$$= \sum_{p\le y,\ p\in Q}\frac{1}{p} - \sum_{p\le y}\frac{1}{p} - \sum_{m\ge 2,\ p\in Q}\frac{1}{mp^m} + O\left(\frac{1}{y}\right)$$
$$= \int_{3/2}^{y}\frac{d\pi(t,Q)}{t} - \log\log y + D(Q) + O\left(\frac{1}{\log y}\right)$$

with some constant $D(Q)$. If we use (1.1) and integration by parts to evaluate the above integral, then we obtain (3.9) by exponentiating (3.10).

4 Proof of Theorem 1 and Theorem 2

We pass now to the proof of Theorem 1. A detailed proof will be given only for the case $k=2$, and it will be indicated how to treat the general case, which is merely technically more complicated than the case $k=2$. In evaluating

$$S(x) := \sum_{n\le x,\ (n,Q)>1}\frac{P_2(n,Q)}{P(n,Q)}$$

first note that the integers n for which $P(n,Q)=P(n)$ contribute $\ll x/\log x$, which follows from (2.1). If $P(n,Q)<P(n)$ and n contains at least two prime factors $p,q\in Q$, then n may be uniquely written as

$$n = mpqr,\ P(m)\le p\le q;\ p,q\in Q,\ (r,Q)=1,\ p(r)>p,\ P(r)=P(n)>q. \tag{4.1}$$

Hence we have

$$S(x) = \sum_{\substack{mpqr\le x,\ P(m)\le p\le q;\ p,q\in Q\\ (r,Q)=1,\ p(r)>p,\ P(r)>q}}\frac{p}{q} + O\left(\frac{x}{\log x}\right). \tag{4.2}$$

Trivially in (4.2) we need to consider only p and q for which

$$\frac{p}{q}\ge\frac{1}{\log x}, \tag{4.3}$$

and we shall show now that we may consider only those m for which

$$m\le e^{(\log x)^\alpha}\qquad(\delta<\alpha<1). \tag{4.4}$$

Namely from Lemma 3 and Lemma 2 (with $\xi = 2$) we have

$$\begin{aligned}\sum_{\substack{mpqr\le x,\ P(m)\le p\le q;\ p,q\in Q\\ (n,Q)=1,\ p(r)>p,\ P(r)>q,\ m\ge e^{\log^\alpha x}}} \frac{p}{q} &\le \sum_{e^{(\log x)^\alpha}\le m\le x}\ \sum_{P(m)\le p\le q} \frac{p}{q} \sum_{r\le \frac{x}{mpq},\ p(r)>p} 1 \\ &\ll x \sum_{e^{(\log x)^\alpha}\le m\le x} \frac{1}{m} \sum_{p\ge P(m)} \frac{1}{\log p} \sum_{q\ge p} \frac{1}{q^2} \\ &\ll x \sum_{m\ge e^{(\log x)^\alpha}} \frac{1}{m(\log P(m))^2} \ll \frac{x}{(\log x)^\alpha},\end{aligned}$$

which is absorbed by the error term in Theorem 1 if $\delta < \alpha < 1$. Likewise we may suppose that

$$p \le e^{(\log x)^\alpha} \qquad (\delta < \alpha < 1), \tag{4.5}$$

since by the preceding argument we obtain

$$\sum_{\substack{mpqr\le x,\ P(m)\le p\le q;\ p,q\in Q\\ (r,Q)=1,\ p(r)>p,\ P(r)>q\\ m\le e^{(\log x)^\alpha},\ p>e^{(\log x)^\alpha}}} \frac{p}{q} \ll x \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{p>e^{(\log x)^\alpha}} \frac{1}{p\log^2 p} \ll \frac{x}{(\log x)^\alpha}.$$

In the portion of the sum in (4.2) for which the conditions (4.3)-(4.5) hold we have

$$mpq \le mp^2 \log x \le e^{3(\log x)^\alpha} \log x,$$

hence for these m, p and q we have

$$\frac{1}{\log\left(\frac{x}{mpq}\right)} = \frac{1}{\log x}\left(1 + O\left(\frac{\log(mpq)}{\log x}\right)\right) = \frac{1}{\log x}\left(1 + O\left((\log x)^{\alpha-1}\right)\right). \tag{4.6}$$

Further the condition $P(r) > q$ may be omitted, since by using Lemma 1 we obtain

$$\begin{aligned}&\sum_{m\le e^{(\log x)^\alpha}}\ \sum_{P(m)\le p\le e^{(\log x)^\alpha}} p \sum_{p\le q\le p\log x} \frac{1}{q} \sum_{r\le \frac{x}{mpq},\ P(r)\le q} 1 \\ &\ll x \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{P(m)\le p\le e^{(\log x)^\alpha}}\ \sum_{p\le q\le p\log x} \frac{1}{q^2} e^{-\frac{\log(x/mpq)}{2\log q}},\end{aligned} \tag{4.7}$$

and by (4.6)

$$\frac{\log(x/mpq)}{\log q} \ge \frac{1}{2}\frac{\log x}{\log q} \ge \frac{1}{2}\frac{\log x}{\log p + \log\log x} \ge \frac{1}{3}(\log x)^{1-\alpha}.$$

Hence the contribution of the left-hand side of (4.7) is

$$\ll x \exp\left(-\frac{1}{6}(\log x)^{1-\alpha}\right) \sum_{m\le e^{(\log x)^\alpha}} \frac{1}{m} \sum_{P(m)\le p\le e^{(\log x)^\alpha}} \frac{1}{p\log p} \ll x\exp\left(-\frac{1}{10}(\log x)^{1-\alpha}\right),$$

which is negligible. Finally by using Lemma 5 and (4.6) we obtain
(4.8)

$$\begin{aligned}
S(x) &= \sum_{m\le e^{(\log x)^\alpha}} \sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha}\\ p\in Q}} p \sum_{\substack{p\le q\le p\log x\\ p\in Q}} \frac{1}{q} \sum_{\substack{r\le \frac{x}{mpq}\\ P(r)\le q}} 1 \sum_{\substack{r\le x/(mpq)\\ (r,Q)=1,\ p(r)>p}} 1 + O\left(\frac{x}{(\log x)^\alpha}\right)\\
&= \left(C(Q)+O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}\sum_{m\le e^{(\log x)^\alpha}}\frac{1}{m}\sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha}\\ p\in Q}}\ell_Q(p)\sum_{\substack{p\le q\le p\log x\\ q\in Q}}\frac{1}{q^2}+O\left(\frac{x}{(\log x)^\alpha}\right)\\
&= \left(C(Q)+O\left(\frac{1}{\log\log x}\right)\right)C'(\delta)\frac{x}{(\log x)^\delta},
\end{aligned}$$

where

$$C'(\delta)=\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\substack{p\ge P(m)\\ p\in Q}}\ell_Q(p)\sum_{\substack{q\ge p\\ q\in Q}}\frac{1}{q^2}.$$

This follows since, by using Lemma 2 and Lemma 6, we obtain,

$$\begin{aligned}
&\sum_{m\le e^{(\log x)^\alpha}}\frac{1}{m}\sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha}\\ p\in Q}}\ell_Q(p)\sum_{\substack{p\le q\le p\log x\\ q\in Q}}\frac{1}{q^2}\\
&=\sum_{m\le e^{(\log x)^\alpha}}\frac{1}{m}\sum_{\substack{P(m)\le p\le e^{(\log x)^\alpha}\\ p\in Q}}\ell_Q(p)\sum_{\substack{q\ge p\\ q\in Q}}\frac{1}{q^2}+O\left(\frac{1}{\log x}\right)\\
&=\sum_{m\le e^{(\log x)^\alpha}}\frac{1}{m}\sum_{\substack{p\ge P(m)\\ p\in Q}}\ell_Q(p)\sum_{\substack{q\ge p\\ q\in Q}}\frac{1}{q^2}+O\left(\sum_{m\le e^{\log^\alpha x}}\frac{1}{m}\sum_{p>e^{\log^\alpha x}}\frac{1}{p(\log p)^{2-\delta}}\right)+O\left(\frac{1}{\log x}\right)\\
&=C'(\delta)+O\left(\sum_{m>e^{(\log x)^\alpha}}\frac{1}{m}\sum_{p\ge P(m)}\frac{1}{(\log p)^{1-\delta}}\sum_{q\ge p}\frac{1}{q^2}\right)+O\left((\log x)^{\alpha(\delta-1)}\right)\\
&=C'(\delta)+O\left(\sum_{m>e^{(\log x)^\alpha}}\frac{1}{m(\log P(m))^{2-\delta}}\right)+O\left((\log x)^{\alpha(\delta-1)}\right)\\
&=C'(\delta)+O\left((\log x)^{\alpha(\delta-1)}\right).
\end{aligned}$$

Theorem 1 for $k=2$ follows then from (4.8) with $C_2(\delta)=C(Q)C'(\delta)$.

To treat the general sum

$$S_k(x):=\sum_{n\le x,\ (n,Q)>1}\frac{P_k(n,Q)}{P(n,Q)}\qquad (k\ge 3,\ \Omega(n,Q)\ge k)$$

one proceeds similarly as in the case $k=2$. Again we may suppose that $P(n,Q)<P(n)$ in view of (2.1). If n has at least k prime factors from Q (otherwise $P_k(n,Q)=0$ by definition) then we may write n uniquely as
(4.9)

$$n=mp_1p_2\ldots p_kr,\ P(m)\le p_1\le\ldots\le p_k;\ p_1\in Q,\ldots,p_k\in Q;\ (r,Q)=1,\ p(r)>p_1,P(r)>p_k.$$

Furthermore we may suppose that

$$\frac{p_1}{p_k} \geq \frac{1}{\log x}, \quad m \leq e^{\log^\alpha x}, \quad p_1 \leq e^{\log^\alpha x} \qquad (\delta < \alpha < 1), \tag{4.10}$$

using the same arguments that were used in the case $k = 2$. Likewise the condition $P(r) > p_k$ may be discarded, and from (4.9) and (4.10) we shall obtain

$$\begin{aligned} S_k(x) &= O\left(\frac{x}{(\log x)^\alpha}\right) \\ &\quad + \sum_{m \leq e^{\log^\alpha x}} \sum_{\substack{P(m) \leq p_1 \leq e^{\log^\alpha x} \\ p_1 \in Q}} p_1 \sum_{\substack{p_1 \leq p_2 \leq p_1 \log x \\ p_2 \in Q}} \cdots \sum_{\substack{p_{k-1} \leq p_k \leq p_1 \log x \\ p_k \in Q}} \frac{1}{p_k} \sum_{\substack{r \leq x/(mp_1 \ldots p_k) \\ (r,Q)=1,\ p(r) > p_1}} 1. \end{aligned}$$

To evaluate the innermost sum we apply Lemma 5 with x replaced by $x/(mp_1 \ldots p_k)$ and $y = p_1$, which is possible in view of (4.10) and $p_1 \leq p_2 \leq \ldots \leq p_k$. The ensuing estimations are performed as in the case $k = 2$. Theorem 1, with $C_k(\delta)$ given by (2.3), readily follows.

We turn now to Theorem 2. We shall prove only (2.6), since the proof of (2.5) is quite similar. If $U(x)$ is the sum appearing in (2.6), then

$$U(x) = \sum_{n \leq x,\ (n,Q) > 1} 1 + \sum_{n \leq x,\ (n,Q) > 1} \frac{B(n,Q) - P(n,Q)}{P(n,Q)} = U_1(x) + U_2(x),$$

say. By using Lemma 4 we immediately obtain

$$U_1(x) = [x] - \sum_{n \leq x,\ (n,Q) = 1} 1 = x - \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta},$$

so it remains to evaluate $U_2(x)$. From (2.1) we have

$$\sum_{\substack{n \leq x,\ (n,Q) > 1 \\ P(n,Q) = P(n)}} \frac{B(n,Q) - P(n,Q)}{P(n,Q)} \leq \sum_{2 \leq n \leq x,\ (n,Q) > 1} \frac{P_2(n,Q) + \Omega(n,Q) P_3(n,Q)}{P(n)} \ll \frac{x}{\log x},$$

since for $n \geq 2$ and $k \geq 1$

$$P_k(n,Q) \leq P_k(n), \quad \Omega(n,Q) \leq \Omega(n) \leq \frac{\log n}{\log 2}.$$

For the remaining n counted by $U_2(x)$ the decomposition (4.1) holds, since the sum is non-zero only if n has at least two prime factors from Q. Thus we have

$$U_2(x) = \sum_{\substack{mpqr \leq x \\ P(m) \leq p \leq q;\ p,q \in Q \\ (r,Q)=1,\ p(r) > p,\ P(r) > q}} \frac{p + B(m,Q)}{q} + O\left(\frac{x}{\log x}\right), \tag{4.11}$$

since

$$B(n,Q)-P(n,Q)=B(mpq,Q)-q=p+B(m,Q)$$

by the additivity of $B(n,Q)$ (see (1.6)). We may suppose that the condition (4.3) holds, since

$$\begin{aligned}\sum_{\substack{mpqr\le x,\ p,q\in Q\\ P(m)\le p\le q,\ p/q\le 1/\log x\\ (r,Q)=1,\ p(r)>p,\ P(r)>q}}\frac{p+B(m,Q)}{q} &\le \sum_{\substack{mpqr\le x,\ p,q\in Q\\ P(m)\le p\le q,\ p/q\le 1/\log x\\ (r,Q)=1,\ p(r)>p,\ P(r)>q}}\frac{p(1+\Omega(m))}{q}\\ &\le \frac{1}{\log x}\sum_{n\le x}\Omega(n)\ll\frac{x\log\log x}{\log x}.\end{aligned}$$

We may also assume that (4.4) holds. Namely we have, for any fixed $c>1$, that the contribution of the sum in (4.11) for which $m>e^{\log^\alpha x}$ is

$$\begin{aligned}&\ll \sum_{e^{\log^\alpha x}<m\le x}\Omega(m)\sum_{p\ge P(m)}p\sum_{q\ge p}\frac{1}{q}\sum_{r\le\frac{x}{mpq},\ (r,Q)=1,\ p(r)>p}1\\ &\ll x\sum_{e^{\log^\alpha x}<m\le x}\frac{\Omega(m)}{m}\sum_{p\ge P(m)}\frac{1}{\log p}\sum_{q\ge p}\frac{1}{q^2}\ll x\sum_{e^{\log^\alpha x}<m\le x}\frac{\Omega(m)}{m\log^2 P(m)}\\ &= x\sum_{e^{\log^\alpha x}<m\le x}\frac{\Omega(m)}{m^{1/c}}\cdot\frac{1}{m^{(c-1)/c}\log^2 P(m)}\ll x\left(\sum_{m\le x}\frac{\Omega^c(m)}{m}\right)^{1/c}\left(\sum_{m>e^{\log^\alpha x}}\frac{1}{m(\log P(m))^{\frac{2c}{c-1}}}\right)^{\frac{c-1}{c}}\\ &\ll x(\log x)^{\frac{1-\alpha-\alpha c}{c}}\log\log x.\end{aligned}$$

Here we used Lemma 3, Hölder's inequality, Lemma 2 with $\xi=\frac{2c}{c-1}$, and the elementary estimate

$$\sum_{n\le x}\Omega^c(n)\ll_c x(\log\log x)^c$$

if c is an integer. Since $\alpha<\delta<1$ and

$$\lim_{c\to\infty}\frac{1-\alpha-\alpha c}{c}=-\alpha,$$

it follows that the above contribution is certainly $\ll x(\log^\delta x\log\log x)^{-1}$ if c is a sufficiently large integer. Similarly we may assume that (4.5) holds, since the contribution of $p>e^{\log^\alpha x}$ is

$$\ll x\sum_{m\le e^{\log^\alpha x}}\frac{\Omega(m)}{m}\sum_{p\ge e^{\log^\alpha x}}\frac{1}{p\log^2 p}\ll x\log^\alpha x\log\log x\cdot(\log x)^{-2\alpha}=\frac{x\log\log x}{(\log x)^\alpha},$$

and that the condition $P(r)>q$ may be omitted. Thus following the method of proof of Theorem 1 we obtain, by using Lemma 5,

$$U_2(x) = \sum_{m\le e^{\log^\alpha x}}\sum_{\substack{P(m)\le p\le e^{\log^\alpha x}\\ p\in Q}}(p+B(m,Q))\sum_{\substack{p\le q\le p\log x\\ q\in Q}}\frac{1}{q}\sum_{\substack{r\le x/(mpq)\\ (r,Q)=1,\ p(r)>q}}1+O\left(\frac{x}{(\log x)^\delta\log\log x}\right)$$

$$= \frac{C(Q)x}{(\log x)^\delta} \sum_{m \le e^{\log^\alpha x}} \sum_{\substack{P(m) \le p \le e^{\log^\alpha x} \\ p \in Q}} (p + B(m,Q))\ell_Q(p) \sum_{\substack{p \le q \le p \log x \\ q \in Q}} \frac{1}{q^2} + O\left(\frac{x}{(\log x)^\delta \log\log x}\right)$$

$$= \left(D_2'(\delta) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta},$$

where

$$D_2'(\delta) = C(Q) \sum_{m=1}^{\infty} \sum_{p \ge P(m),\ p \in Q} (p + B(m,Q))\ell_Q(p) \sum_{q \ge p,\ q \in Q} \frac{1}{q^2}. \tag{4.12}$$

Hence we obtain

$$U(x) = x + \left(D_2'(\delta) - C(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta},$$

which proves (2.6) with $D_2(\delta) = D_2'(\delta) - C(Q)$. In proving (2.5) we shall encounter $\omega(m)$ instead of $\Omega(m)$, which is harmless since $\omega(m) \le \Omega(m)$. The only change is that, as $\beta(n,Q)$ counts the sum of distinct prime factors of n which belong to Q, in the analogue of (4.11) we shall suppose that $P(m) < p < q$, as the cases when $P(m) = p$, $p = q$ will make a negligible contribution. Hence the constant analogous to $D_2'(\delta)$ of (4.12) will be

$$D_1'(\delta) = C(Q) \sum_{m=1}^{\infty} \sum_{p > P(m),\ p \in Q} (p + \beta(m,Q))\ell_Q(p) \sum_{q > p,\ q \in Q} \frac{1}{q^2},$$

which will clearly give $0 < D_1(\delta) < D_2(\delta)$ in Theorem 2. The essential reason why the method of proof of Theorem 1 could be extended to yield Theorem 2 is that one encounters $\Omega(m)$ at various places in the estimations (coming from $B(m,Q) \le \Omega(m)P(m,Q)$). Since $\Omega(m)$ has average and normal order equal to $\log\log m$, all the estimates are only affected by this factor which is small and therefore harmless.

5 Proof of Theorem 3

We shall prove (2.7) only, since the proof of (2.8) is similar. We have

$$\sum_{n \le x,\ (n,Q) > 1} \frac{1}{\beta(n,Q)} = \sum_{\substack{n \le x,\ (n,Q)>1 \\ P(n,Q) < P(n)}} \frac{1}{\beta(n,Q)} + \sum_{\substack{n \le x,\ (n,Q)>1 \\ P(n,Q) = P(n)}} \frac{1}{\beta(n,Q)} \tag{5.1}$$

$$= \sum_{\substack{n \le x,\ (n,Q)>1 \\ P(n,Q) < P(n)}} \frac{1}{\beta(n,Q)} + O\left(\frac{x}{\log x}\right)$$

$$= T(x) + O\left(\frac{x}{\log x}\right),$$

say, since (2.9) holds with $P(n)$ in place of $\beta(n)$. If n is counted by $T(x)$, then n can be written uniquely as

$$n = mr,\ (r,Q) = 1,\ p(r) > P(m) \in Q. \tag{5.2}$$

Therefore we have

$$T(x) = \sum_{m\le x,\ P(m)\in Q} \frac{1}{\beta(m,Q)} \sum_{r\le x/m,\ (r,Q)=1,\ p(r)>P(m)} 1 = T_1(x) + T_2(x), \tag{5.3}$$

say, where in $T_1(x)$ we have $m \le e^{\log^\alpha x}$ $(\delta < \alpha < 1)$, and in $T_2(x)$ we have $e^{\log^\alpha x} < m \le x$. By using Lemma 3, Lemma 2 and

$$\beta(m,Q) \ge P(m) \ge \log P(m)$$

if $P(m) \in Q$, we obtain that
(5.4)

$$T_2(x) \ll x \sum_{m>e^{\log^\alpha x},\ P(m)\in Q} \frac{1}{m\beta(m,Q)\log P(m)} \ll x \sum_{m>e^{\log^\alpha x}} \frac{1}{m\log^2 P(m)} \ll \frac{x}{(\log x)^\alpha}.$$

Now applying Lemma 5 we have

$$\begin{aligned} T_1(x) &= \sum_{m\le e^{\log^\alpha x},\ P(m)\in Q} \frac{1}{\beta(m,Q)} \sum_{r\le x/m,\ (r,Q)=1,\ p(r)>P(m)} 1 \\ &= \frac{C(Q)x}{(\log x)^\delta}\left(1 + O\left(\frac{1}{\log\log x}\right)\right) \sum_{m\le e^{\log^\alpha x},\ P(m)\in Q} \frac{\ell_Q(P(m))}{m\beta(m,Q)} \\ &= \left\{C(Q) \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\beta(m,Q)} + O\left(\frac{1}{\log\log x}\right)\right\} \frac{x}{(\log x)^\delta} \end{aligned} \tag{5.5}$$

by repeating the argument used in the estimation of $T_1(x)$. Thus we obtain from (5.1), (5.3), (5.4) and (5.5) that (2.7) holds with

$$\eta_1(Q) = C(Q) \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\beta(m,Q)}.$$

Likewise we obtain (2.8) with

$$\eta_2(Q) = C(Q) \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{mB(m,Q)},$$

and $0 < \eta_2(Q) < \eta_1(Q)$ holds in view of (1.10).

6 Proof of Theorem 4 and Theorem 5

If we can establish, for $x \ge 2$,

$$\sum_{2\le n\le x}\frac{1}{\log P(n)} = \frac{e^\gamma x}{\log x} + R(x), \qquad R(x) = O\left(\frac{x}{\log^2 x}\right), \tag{6.1}$$

then by partial summation (6.1) readily implies (2.10) with

$$B = \int_2^\infty R(t)\frac{dt}{t^2} - e^\gamma \log\log 2.$$

To prove (6.1) note that

$$\begin{aligned}\sum_{2\le n\le x}\frac{1}{\log P(n)} &= \sum_{p\le x}\frac{1}{\log p}\psi(\frac{x}{p},p)\\ &= \left(\sum_{p\le L}+\sum_{L<p\le\sqrt{x}}+\sum_{\sqrt{x}<p\le x}\right)\frac{1}{\log p}\psi(\frac{x}{p},p) = S_1+S_2+S_3,\end{aligned}$$

say, where

$$L := \exp\left(\frac{\log x}{(\log\log x)^2}\right).$$

From (3.1) of Lemma 1 we have

$$S_1 \ll x\sum_{p\le L}\frac{1}{p\log p}e^{-\frac{\log x}{2\log p}} \le xe^{-\frac{1}{2}(\log\log x)^2}\sum_p \frac{1}{p\log p} \ll \frac{x}{\log^2 x}, \tag{6.2}$$

since $\sum_p 1/(p\log p)$ converges. In the range $L < p \le \sqrt{x}$ in S_2 we may use the asymptotic formula (3.2) to evaluate $\psi(\frac{x}{p},p)$. We obtain

$$S_2 = x\sum_{L<p\le\sqrt{x}}\frac{1}{p\log p}\rho\left(\frac{\log x}{\log p}-1\right) + O\left(x\sum_{L<p\le\sqrt{x}}\frac{\log\left(\frac{\log x}{\log p}+1\right)}{p\log^2 p}\rho\left(\frac{\log x}{\log p}-1\right)\right). \tag{6.3}$$

By using (3.3) and (3.4) it is seen that the contribution of the O-term in (6.3) is

$$\begin{aligned} &\ll x\sum_{L<p\le\sqrt{x}}\frac{\log\left(\frac{\log x}{\log p}+1\right)}{p\log^2 p}e^{-\left(\frac{\log x}{\log p}-1\right)\log\left(\frac{\log x}{\log p}-1\right)}\\ &= x\int_{L+0}^{\sqrt{x}}\frac{\log\left(\frac{\log x}{\log t}+1\right)}{t\log^2 t}e^{-\left(\frac{\log x}{\log t}-1\right)\log\left(\frac{\log x}{\log t}-1\right)}d\pi(t)\\ &= x\int_{L}^{\sqrt{x}}\frac{\log\left(\frac{\log x}{\log t}+1\right)}{t\log^3 t}e^{-\left(\frac{\log x}{\log t}-1\right)\log\left(\frac{\log x}{\log t}-1\right)}dt + O\left(\frac{x}{\log^2 x}\right)\\ &= \frac{x}{\log^2 x}\int_2^{(\log\log x)^2} u\log(u+1)e^{-(u-1)\log(u-1)}du + O\left(\frac{x}{\log^2 x}\right) \ll \frac{x}{\log^2 x},\end{aligned}$$

after the substitution $\frac{\log x}{\log t} = u$. Similarly, substituting $\frac{\log x}{\log t} - 1 = v$, the main term in (6.3) equals

$$
\begin{aligned}
x \int_{L+0}^{\sqrt{x}} \frac{1}{t \log t} \rho \left(\frac{\log x}{\log t} - 1 \right) d\pi(t) &= x \int_{L}^{\sqrt{x}} \rho \left(\frac{\log x}{\log t} - 1 \right) \frac{dt}{t \log^2 t} \\
&\quad - x \int_{L+0}^{\sqrt{x}} \Delta(t) \, d \left(\frac{1}{t \log t} \rho \left(\frac{\log x}{\log t} - 1 \right) \right) + O\left(\frac{x}{\log^2 x} \right) \\
&= \frac{x}{\log x} \int_{1}^{(\log\log x)^2 - 1} \rho(v) \, dv + O\left(\frac{x}{\log^2 x} \right) \\
&= \frac{(e^\gamma - 1)x}{\log x} + O\left(\frac{x}{\log^2 x} \right)
\end{aligned}
$$

Here we used (3.4), $\rho(u) = 1$ for $0 \le u \le 1$, $\rho'(u) = -\frac{\rho(u-1)}{u} \ll e^{-u}$ and

$$
\int_0^\infty \rho(v) \, dv = e^\gamma. \tag{6.4}
$$

For a proof of the well-known relation (6.4), see e.g. G. Tenenbaum [13]. Incidentally (6.4) follows in an elementary way if we compare our proof of Theorem 3 with the elementary derivation of Theorem 1.2 of De Koninck - Sitaramachandrarao [6]. Hence we obtain

$$
S_1 + S_2 = \frac{(e^\gamma - 1)x}{\log x} + O\left(\frac{x}{\log^2 x} \right). \tag{6.5}
$$

Lastly we have, since $\psi(x, y) = [x]$ for $y \ge x$,

$$
S_3 = \sum_{\sqrt{x} < p \le x} \frac{1}{\log p} \psi(\frac{x}{p}, p) = \sum_{\sqrt{x} < p \le x} \frac{1}{\log p} \left[\frac{x}{p} \right] = \sum_{m \le \sqrt{x}} \sum_{\sqrt{x} < p \le x/m} \frac{1}{\log p}.
$$

From the prime number theorem we obtain

$$
\sum_{p \le y} \frac{1}{\log p} = \frac{y}{\log^2 y} + O\left(\frac{y}{\log^3 y} \right),
$$

which yields

$$
\begin{aligned}
S_3 &= \sum_{m \le \sqrt{x}} \left(\frac{x}{m \log^2 \left(\frac{x}{m} \right)} + O\left(\frac{x}{m \log^3 \left(\frac{x}{m} \right)} \right) + O\left(\frac{\sqrt{x}}{\log^2 x} \right) \right) \\
&= \sum_{m \le \sqrt{x}} \frac{x}{m \log^2 \left(\frac{x}{m} \right)} + O\left(\frac{x}{\log^2 x} \right) = x \int_1^{\sqrt{x}} \frac{dt}{t \log^2 \left(\frac{x}{t} \right)} + O\left(\frac{x}{\log^2 x} \right) \\
&= x \int_{\sqrt{x}}^{x} \frac{du}{u \log^2 u} + O\left(\frac{x}{\log^2 x} \right) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x} \right).
\end{aligned} \tag{6.6}
$$

The asymptotic formula (6.1) follows then from (6.5) and (6.6).

To prove Theorem 4 we shall prove

$$\sum_{n\le x,\ (n,Q)>1} \frac{1}{\log P(n,Q)} = \left(D(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}, \tag{6.7}$$

where

$$D(Q) = C(Q)\sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\log P(m)}, \tag{6.8}$$

and $C(Q)$, $\ell_Q(y)$ are as in Lemma 5. By partial summation Theorem 4 follows from (6.7) with $F(Q) = D(Q)/(1-\delta)$. Let $\delta < \alpha < 1$. Then we have

$$\begin{aligned}\sum_{n\le x,\ (n,Q)>1} \frac{1}{\log P(n,Q)} &= \sum_{\substack{n\le x,\ (n,Q)>1\\ P(n,Q)<P(n)\\ P(n,Q)\le e^{\log^\alpha x}}} \frac{1}{\log P(n,Q)} + O\left(\frac{x}{(\log x)^\alpha}\right)\\ &= \Sigma_0 + O\left(\frac{x}{(\log x)^\alpha}\right),\end{aligned}$$

say. Here we used the bound

$$\sum_{\substack{n\le x,\ (n,Q)>1\\ P(n,Q)=P(n)}} \frac{1}{\log P(n,Q)} \le \sum_{2\le n\le x}\frac{1}{\log P(n)} \ll \frac{x}{\log x},$$

which is a trivial consequence of (6.1). If n is counted by $\sum_0$, then n can be uniquely written as

$$n = mr,\ P(m)\in Q,\ (r,Q)=1,\ p(r) > P(m),$$

since $P(m) = P(n,Q) < P(n) = p(r)$. Thus

$$\Sigma_0 = \sum_{\substack{m\le x,\ P(m)\in Q\\ P(m)\le e^{\log^\alpha x}}} \frac{1}{\log P(m)} \sum_{\substack{r\le x/m,\ (r,Q)=1\\ p(r)>P(m)}} 1 = \Sigma_1 + \Sigma_2,$$

say, where in Σ_1 we have $m \le K := e^{\log^\beta x}$ $(\alpha < \beta < 1)$, and in Σ_2 we have $K < m \le x$. By (3.1) of Lemma 1 we obtain

$$\begin{aligned}\Sigma_2 &\le x\sum_{\substack{K<m\le x\\ P(m)\le e^{\log^\alpha x}}} \frac{1}{m\log P(m)} = x\sum_{p\le e^{\log^\alpha x}} \frac{1}{p\log p}\sum_{\substack{\frac{K}{p}<n\le\frac{x}{p}\\ P(n)\le p}}\frac{1}{n}\\ &= x\sum_{p\le e^{\log^\alpha x}}\frac{1}{p\log p}\left(\frac{\psi(t,p)}{t}\Big|_{\frac{K}{p}}^{\frac{x}{p}} + \int_{\frac{K}{p}}^{\frac{x}{p}}\psi(t,p)\,\frac{dt}{t^2}\right)\\ &\ll x\sum_{p\le e^{\log^\alpha x}}\frac{1}{p\log p}\left(e^{-\frac{\log K}{2\log p}} + \int_{\frac{K}{p}}^{\frac{x}{p}} e^{-\frac{\log t}{2\log p}}\frac{dt}{t}\right)\\ &\ll x\sum_{p\le e^{\log^\alpha x}}\frac{1}{p}e^{-\frac{\log K}{2\log p}} \ll xe^{-\frac{1}{2}\log^{\beta-\alpha} x}\sum_{p\le e^{\log^\alpha x}}\frac{1}{p} \ll \frac{x}{(\log x)^\delta\log\log x},\end{aligned}$$

since $\beta > \alpha$ and $\sum_{p\le x} \frac{1}{p} \ll \log\log x$. In Σ_1 we evaluate the inner sum by applying Lemma 5 to obtain

$$\begin{aligned} S_1 &= \left(C(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta} \sum_{\substack{m\le e^{\log^\beta x},\ P(m)\in Q \\ P(m)\le e^{\log^\alpha x}}} \frac{\ell_Q(P(m))}{m\log P(m)} \\ &= \left(D(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta}, \end{aligned}$$

where $D(Q)$ is given by (6.8). This proves (6.7), but to justify the last equality above we proceed as follows. From Lemma 6 and Lemma 2 we obtain

$$\sum_{m>X,\ P(m)\in Q} \frac{\ell_Q(P(m))}{m\log P(m)} \ll \sum_{m>X} \frac{1}{m\log^{2-\delta} P(m)} \ll \frac{1}{\log^{1-\delta} X}. \tag{6.9}$$

Hence setting

$$Y = \exp\left((\log\log x)^{\frac{1}{1-\delta}}\right)$$

and using (6.9) we have

$$\begin{aligned} &\sum_{\substack{m\le e^{\log^\beta x},\ P(m)\in Q \\ P(m)\le e^{\log^\alpha x}}} \frac{\ell_Q(P(m))}{m\log P(m)} \\ &= \sum_{\substack{m\le Y,\ P(m)\in Q \\ P(m)\le e^{\log^\alpha x}}} \frac{\ell_Q(P(m))}{m\log P(m)} + O\left(\frac{1}{\log^{1-\delta} Y}\right) \\ &= \sum_{m\le Y,\ P(m)\in Q} \frac{\ell_Q(P(m))}{m\log P(m)} + O\left(\frac{1}{\log\log x}\right) \\ &= \sum_{m=2,\ P(m)\in Q}^{\infty} \frac{\ell_Q(P(m))}{m\log P(m)} + O\left(\frac{1}{\log\log x}\right), \end{aligned}$$

since $P(m) > e^{\log^\alpha x}$ is impossible if $m \le Y$. This completes the proof of Theorem 4.

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