Arithmetic characterization of regularly varying functions

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Riassunto. Si investigano formule asintotiche che riguardano somme aritmetiche con funzioni à oscillazione lenta. Parecchi teoremi, tipo d'Abel e Tauber, vengono provati con metodi elementari e analitici.

Key Words: Regularly and slowly varying functions, arithmetic sums, Abelian and Tauberian theorems.

1 Introduction

Given a function $f : \mathbf{R}^+ \to \mathbf{R}^+$, one is often interested in obtaining its average value on the positive integers, say by estimating $\frac{1}{x}\sum_{n\le x} f(n)$. It is common to obtain that, as $x \to \infty$,

$$\sum_{n\le x} f(n) = (C + o(1))xf(x). \tag{1.1}$$

However, this formula certainly does not hold for all functions $f : \mathbf{R}^+ \to \mathbf{R}^+$. It does however hold for a large class of so-called *"regularly varying*

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functions". A regularly varying function f is a continuous function for which there exists $\rho > 0$ (called the index of f) such that

$$\lim_{x\to\infty} \frac{f(cx)}{f(x)} = x^\rho, \quad \text{for all } c > 0. \tag{1.2}$$

We shall denote the set of all regularly varying functions by $\Re$. We shall also denote by $\mathcal{L}$ the set of *slowly varying functions*, namely those functions in $\Re$ for which the index $\rho = 0$. It is easy to show that if $f \in \Re$, then there exists $L \in \mathcal{L}$ such that $f(x) = x^\rho L(x)$, with ρ being the index of f.

As is shown in Theorem 1 below, if $f \in \Re$ with index $\rho > -1$, then (1.1) is satisfied with $C = \frac{1}{\rho+1}$.

Given an arithmetical function g defined by

$$g(n) = \sum_{p|n} f(p),$$

where $f \in \Re$ with index $\rho > 0$, one can show (see Theorem 2 below) that

$$\sum_{n\le x} g(n) = (1 + o(1))Cx\frac{f(x)}{\log x}, \tag{1.3}$$

with $C = \dfrac{\zeta(1+\rho)}{1+\rho}$.

Theorems 1 and 2 stated in section 2 are called *abelian theorems*.

Is the converse of these theorems true? In other words, given a function f which satisfies (1.1) for a certain constant $C > 0$, is it true that $f \in \Re$? And given an arithmetical function $g(n) = \sum_{p|n} f(p)$ satisfying (1.3), is it true that $f \in \Re$? These kinds of results are called *tauberian theorems*. We shall prove that, under certain conditions, each of these two tauberian results is true. The first of these is contained in Theorem 3, whose proof is given in section 3. The other result is Theorem 4, which is really a consequence of Theorem 5 and Theorem 6. These two results, which seem to be of independent interest, are of a fairly general nature. As is usually the case, tauberian theorems are more difficult than the corresponding abelian theorems. They involve a certain so-called *tauberian condition*, which enables one to make the appropriate deduction. In our case this will be, crudely speaking, the property that the function in question is nondecreasing.

In what follows we have used, whenever possible, standard notation. In particular, the letters $X, X_0, X_1, \ldots$ denote large positive constants, not necessarily the same ones at each occurence.

2 Abelian theorems

Theorem 1. *Let $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f \in \Re$ with index $\rho > -1$. Then, as $x \to \infty$, (1.1) holds with $C = \frac{1}{\rho+1}$.*

Proof. Consider first the case $\rho > 0$ and let $f(x) = x^\rho L(x)$, where $L \in \mathcal{L}$. Because L is slowly varying, it is known (see Seneta [4]) that there exist two functions $K(x)$ and $\eta(x)$ and two real numbers $K > 0$ and $x_0 > 0$ such that

$$L(x) = K(x) e^{\int_{x_0}^x \frac{\eta(t)}{t}\,dt},$$

with $\lim_{x\to\infty} K(x) = K$ and $\lim_{x\to\infty} \eta(x) = 0$. The sum in (1.1) is therefore

$$\sum_{n \le x_0} n^\rho L(n) + \sum_{x_0 < n \le \frac{x}{A}} n^\rho L(n) + \sum_{\frac{x}{A} < n \le x} n^\rho L(n) = S_1 + S_2 + S_3,$$

say, where $A > 0$ is a large constant. Trivially, $S_1 = O(1)$. Set

$$\overline{L}(x) = K e^{\int_{x_0}^x \frac{\eta(t)}{t}\,dt}.$$

Then $\overline{L}(x) \sim L(x)$ as $x \to \infty$, and $\overline{L}(x)$ is differentiable. Since $\rho > 0$, $\eta(x) \to 0$ as $x \to \infty$, we have that

$$\overline{L}\,'(x) = x^{\rho-1}\overline{L}(x)(\rho + \eta(x)) > 0$$

for $x \ge x_1$, which implies that $\overline{L}(x)$ is increasing for $x \ge x_1$. But

$$\sum_{x_1 < n \le x} F(n) = \int_{x_1}^x F(t)\,dt + O(F(x)), \tag{2.1}$$

if $F(x)$ is positive, continuous and increasing for $x \ge x_1$. Therefore with $F(x) = x^\rho \overline{L}(x)$, we obtain

$$\begin{aligned}
S_2 &= O(1) + \sum_{x_1 < n \le \frac{x}{A}} n^\rho L(n) \ll 1 + \sum_{x_1 < n \le \frac{x}{A}} F(n) \\
&= \int_{x_1}^{x/A} F(t)\,dt + O(F(x)) = (1 + o(1)) \left(\frac{x}{A}\right)^{\rho+1} \frac{\overline{L}\left(\frac{x}{A}\right)}{\rho+1} + O\left(\left(\frac{x}{A}\right)^\rho \overline{L}\left(\frac{x}{A}\right)\right) \\
&= (1 + o(1)) \left(\frac{x}{A}\right)^{\rho+1} \frac{L\left(\frac{x}{A}\right)}{\rho+1} + O\left(\left(\frac{x}{A}\right)^\rho \overline{L}\left(\frac{x}{A}\right)\right) \\
&\ll \frac{x^{\rho+1}}{A^{\rho+1}} L(x)
\end{aligned}$$

since $\overline{L}\left(\frac{x}{A}\right) \sim L(x)$ as $x \to \infty$ and for $\alpha > -1$,

$$\int_{x_1}^{x} t^{\alpha} L(t)\, dt \sim \frac{x^{\alpha+1}}{\alpha+1} L(x) \tag{2.2}$$

by a classical result of J. Karamata (see BGT [2], prop. 1.5.8). Now using the Uniform Convergence Theorem (see Bartle [1], p. 67), (2.1) and (2.2), we have

$$\begin{aligned} S_3 &= (1+o(1)) \sum_{\frac{x}{A}<n\le x} n^{\rho}\overline{L}(n) \sim \int_{x/A}^{x} t^{\rho}\overline{L}(t)\, dt \\ &= \left(\int_{x_1}^{x} - \int_{x_1}^{x/A}\right) t^{\rho}\overline{L}(t)\, dt \\ &= (1+o(1))\frac{x^{\rho+1}\overline{L}(x)}{\rho+1} - (1+o(1))\left(\frac{x}{A}\right)^{\rho+1} \frac{\overline{L}\left(\frac{x}{A}\right)}{\rho+1} \\ &= \Big(1+o(1)+O(A^{-\rho-1})\Big)\frac{x^{\rho+1}L(x)}{\rho+1}. \end{aligned}$$

Thus combining the preceeding estimates we obtain

$$\sum_{n\le x} f(n) = \Big(1+o(1)+O(A^{-\rho-1})\Big)\frac{x^{\rho+1}L(x)}{\rho+1}.$$

Since $A>0$ may be chosen arbitrarily large, (1.1) follows for $\rho>0$.

We have treated first the case $\rho>0$, since this condition ensures both $f(x)\to\infty$ as $x\to\infty$ and the monotonicity of $x^{\rho}\overline{L}(x)$. In the case $\rho=0$, it may already happen that $f(x)=L(x)\to\infty$, but $\overline{L}(x)$ is not monotonic. An example is given by

$$L(x)=\overline{L}(x)=\exp\left\{\int_2^x \frac{\eta(t)}{t}\, dt\right\}, \qquad \eta(x)=\frac{1-\sin x^2-1/x}{\log x}.$$

For this reason we reduce the remaining case $-1<\rho\le 0$ to the previous one, by setting $f_0(x)=xf(x)$, so that $f_0\in\Re$ with index $\rho_0=\rho+1>0$; and we may use (1.1) with f replaced by f_0. Partial summation gives then

$$\sum_{n\le x} f(n) = \sum_{n\le x}\frac{f_0(n)}{n} = \frac{1}{x}\left(\sum_{n\le x} f_0(n)\right) + \int_1^x \left(\sum_{n\le t} f_0(n)\right)\frac{dt}{t^2}$$

$$\begin{aligned} &= \frac{1}{x}(1+o(1))\frac{x^{\rho+2}L(x)}{\rho+2} + \int_{X_0}^{x}(1+o(1))\frac{t^{\rho+2}L(t)}{t^2(\rho+2)}\,dt + O(1) \\ &= (1+o(1))\frac{x^{\rho+1}L(x)}{\rho+2} + (1+o(1))\frac{x^{\rho+1}L(x)}{(\rho+2)(\rho+1)} \\ &= (1+o(1))\frac{x^{\rho+1}L(x)}{\rho+1}, \end{aligned}$$

where (2.2) was used. This completes the proof of Theorem 1.

Theorem 2. *Let $f \in \Re$ with index $\rho > 0$, then*

$$\sum_{n\le x}\sum_{p|n} f(p) = (1+o(1))Cx\frac{f(x)}{\log x},$$

with $C = \dfrac{\zeta(1+\rho)}{1+\rho}$.

Proof. See De Koninck – Ivić [3].

3 Tauberian theorems

Theorem 3. *Let $f : [1,+\infty[\to \mathbf{R}^+$ be a continuous function such that $\lim_{x\to\infty} f(x) = +\infty$. Assume that, as $x \to \infty$,*

$$\sum_{n\le x} f(n) = (1+o(1))Cxf(x) \tag{3.1}$$

for some $C > 0$. Then $f \in \Re$ with index $\rho = \frac{1}{C} - 1$ if any of the following conditions is satisfied:

(i) $f(x)$ is nondecreasing for $x \ge x_0$;

(ii) $f'(x)$ is continuous for $x \ge x_1$ and

$$\int_{x_1}^{x}\psi(t)f'(t)\,dt = o\left(\int_{x_1}^{x} f(t)\,dt\right)$$

as $x \to \infty$, where $\psi(x) = x - [x] - \frac{1}{2}$;

(iii) $f'(x)$ is continuous for $x \ge x_1$ and, as $x \to \infty$,

$$f'(x) = o(f(x)).$$

Proof. Suppose (i) holds. Then by (2.1)

$$\sum_{n\le x} f(n) = \int_{x_0}^{x} f(t)\,dt + O(f(x)),$$

whence on comparison with (3.1) we find that

$$\int_{x_0}^{x} f(t)\,dt = (1+o(1))Cxf(x) \qquad (x\to\infty).$$

Then by J. Karamata's well known theorem (see Theorem 1.6.1 of BGT [2]), it follows that $f(x)$ is regularly varying with index $\rho = \frac{1}{C} - 1$.

Since (iii) trivially implies (ii), it suffices to consider (ii). For this we use the familiar Euler-MacLaurin summation formula in the form

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\,dt - \psi(b)f(b) + \psi(a)f(a) + \int_a^b \psi(t)f'(t)\,dt.$$

Using (3.1) and (ii), we obtain

$$\begin{aligned}(C+o(1))xf(x) &= (1+o(1))\int_{x_1}^{x} f(t)\,dt + \int_{x_1}^{x} f'(t)\psi(t)\,dt\\ &= (1+o(1))\int_{x_1}^{x} f(t)\,dt,\end{aligned}$$

whence the conclusion of the theorem follows as in the previous case.

We turn now to the Tauberian converse of (1.3). Our result is

Theorem 4. *Let $f : [2,+\infty[\to \mathbf{R}^+$ be a differentiable function. Assume that there exists a constant $C>0$ such that, as $x\to\infty$,*

$$\sum_{n\le x}\sum_{p|n} f(p) = (1+o(1))Cx\frac{f(x)}{\log x}, \tag{3.2}$$

and denote by ρ the unique positive solution of the equation

$$C = \frac{\zeta(1+\rho)}{1+\rho}. \tag{3.3}$$

If $f(x)/\log x$ is nondecreasing, then $f(x) = x^{\rho+o(1)}$ as $x\to\infty$. If $f(x)/(x^{\rho}\log x)$ is nondecreasing, then $f\in\Re$ with index ρ.

First of all if $f(x)/\log x$ is nondecreasing, so is $f(x)$. Hence by Lemma 2 the asymptotic formula (3.2) may be replaced by

$$\int_2^x \frac{f(v)}{\log v}\left[\frac{x}{v}\right] dv = (1+o(1))Cx\frac{f(x)}{\log x} \qquad (x\to\infty). \tag{3.4}$$

The conclusion that $f(x) = x^{\rho+o(1)}$ follows then, with $F(x) = f(x)/\log x$, from the following

Theorem 5. *Let $F : [2,+\infty[\to \mathbf{R}^+$ be a continuous, nondecreasing function. Assume that there exists a constant $C > 0$ such that, as $x\to\infty$,*

$$\int_2^x F(v)\left[\frac{x}{v}\right] dv = (C+o(1))xF(x), \tag{3.5}$$

and denote by ρ be the unique positive solution of (3.3). Then, as $x\to\infty$,

$$F(x) = x^{\rho+o(1)}. \tag{3.6}$$

Theorem 5 is of a fairly general nature, but it falls short of establishing that $F \in \Re$ with index ρ. If we set $L(x) = F(x)x^{-\rho}$, it is seen (defining appropriately $F(x)$ for $1 \le x \le 2$, if necessary) that (3.5) reduces to

$$\int_1^x L(v)\left[\frac{x}{v}\right] v^\rho\, dv = (C+o(1))xL(x) \qquad (x\to\infty). \tag{3.8}$$

Hence the conclusion of Theorem 5 is equivalent to the assertion that $L(x) = x^{o(1)}$, namely that $x^{-\varepsilon} \ll_\varepsilon L(x) \ll_\varepsilon x^\varepsilon$ for any given $\varepsilon > 0$. However, $L(x) = x^{o(1)}$ is still far from $L \in \mathcal{L}$. Another way to look at (3.8) is to make the change of variable $v = x/u$. Then (3.8) becomes a special case of

$$\int_1^x L(\frac{x}{u})h(u)\, du = (1+o(1))L(x)\int_1^\infty h(u)\, du \qquad (x\to\infty) \tag{3.9}$$

with $h(u) = [u]u^{-\rho-2}$. However, (3.9) holds if $L(x)$ is slowly varying and $h(u)$ is a Riemann integrable function such that $0 < h(u) \ll u^{-c}$ for some $c > 1$. This is not difficult to show directly, and follows for example from Lemma 1 of J.P. Tull [5]. What we need therefore, to deduce from (3.8) that $L \in \mathcal{L}$, is a tauberian converse of (3.9). This is contained in

Theorem 6. *Let* $L : [1, +\infty[\to \mathbf{R}^+$ *be a continuous function, nondecreasing or nonincreasing for sufficiently large* x*. If (3.9) holds for some Riemann integrable function* $h(u)$ *such that* $0 < h(u) \ll u^{-c}$ *for some* $c > 1$*, then* $L \in \mathcal{L}$*.*

Hence from (3.4) and Theorem 6 (with $L(x) = f(x)/(x^\rho \log x)$) the second conclusion of Theorem 4, comes from the use of Lemma 2, and could be perhaps weakened.

Our main task is therefore to prove Theorem 5 and Theorem 6. It will turn out that the proof of the former is the one that is complicated, and necessitates several lemmas. These will be proved in section 4, while the proofs of Theorem 5 and Theorem 6 will be given in section 5.

4 Preliminary results

In this section we shall state and prove the necessary lemmas.

Lemma 1. *Let f be as in Theorem 4 and moreover assume that it is a nondecreasing function. If $0 < \varepsilon < 1$ is given, then for $x \ge X = X(\varepsilon)$,*

$$x^{\frac{1}{C+1}(1-\varepsilon)} \le f(x) \le x^{\frac{1}{C}(1+\varepsilon)}. \tag{4.1}$$

Proof. Clearly (3.2) is equivalent to

$$\sum_{p\le x} f(p)\left[\frac{x}{p}\right] \sim C\frac{xf(x)}{\log x} \qquad (x\to\infty). \tag{4.2}$$

Write

$$\sum_{p\le x} f(p)\left[\frac{x}{p}\right] = x\sum_{p\le x}\frac{f(p)}{p} - \sum_{p\le x} f(p)\left(\frac{x}{p} - \left[\frac{x}{p}\right]\right) = S_1(x) - S_2(x), \tag{4.3}$$

say. Because of (4.3), we have that (4.2) is equivalent to

$$S_1(x) - S_2(x) \sim C\frac{xf(x)}{\log x}. \tag{4.4}$$

Note that, given $\varepsilon_1 > 0$, there exists $X_1 > 0$ such that, if $x \ge X_1$,

$$0 \le S_2(x) \le \sum_{p\le x} f(p) \le f(x)\sum_{p\le x} 1 \le (1+\varepsilon_1)\frac{xf(x)}{\log x}. \tag{4.5}$$

Combining (4.4) and (4.5), we get that, if $x \ge X_2$ $(> X_1)$, then

$$(C-\varepsilon_1)\frac{xf(x)}{\log x} \le S_1(x) \le S_2(x) + (1+\varepsilon_1)C\frac{xf(x)}{\log x},$$

that is,

$$(C-\varepsilon_1)\frac{xf(x)}{\log x} \le S_1(x) \le (1+\varepsilon_1)(C+1)\frac{xf(x)}{\log x}. \tag{4.6}$$

Using the prime number theorem in the form

$$\pi(x) = \int_2^x \frac{dt}{\log t} + R(x),$$

with $R(x) = O\left(xe^{-\sqrt{\log x}}\right)$, we have that

$$\frac{S_1(x)}{x} = \sum_{p \le x} \frac{f(p)}{p} = \int_{2-0}^{x} \frac{f(t)}{t}\, d\pi(t) = \int_{2}^{x} \frac{f(t)}{t\log t}\, dt + \int_{2-0}^{x} \frac{f(t)}{t}\, dR(t);$$

integrating the last integral by parts and using the fact that $R(t) \ll te^{-\sqrt{\log t}}$ and that $f'(t) \ge 0$, it follows that

$$S_1(x) \sim x \int_2^x \frac{f(t)}{t\log t}\, dt.$$

Hence

$$(1-\varepsilon_1)x\int_2^x \frac{f(t)}{t\log t}\, dt \le S_1(x) \le (1+\varepsilon_1)x\int_2^x \frac{f(t)}{t\log t}\, dt, \tag{4.7}$$

provided $x > X_3$ $(> X_2)$. Combining (4.6) and (4.7), we get that

$$(1-\varepsilon_1)x\int_2^x \frac{f(t)}{t\log t}\, dt \le (1+\varepsilon_1)(C+1)\frac{xf(x)}{\log x},$$

that is,

$$\int_2^x \frac{f(t)}{t\log t}\, dt \le \frac{(1+\varepsilon_1)(C+1)}{(1-\varepsilon_1)}\frac{f(x)}{\log x}. \tag{4.8}$$

Similarly, we get that

$$\int_2^x \frac{f(t)}{t\log t}\, dt \ge \frac{(C-\varepsilon_1)}{(1+\varepsilon_1)}\frac{f(x)}{\log x}. \tag{4.9}$$

We may combine (4.8) and (4.9) to get

$$(C-\varepsilon_2)\frac{f(x)}{\log x} \le \hat{f}(x) \le (C+1+\varepsilon_2)\frac{f(x)}{\log x}, \tag{4.10}$$

where

$$\hat{f}(x) \stackrel{\text{def}}{=} \int_2^x \frac{f(t)}{t\log t}\, dt.$$

Using (4.10), we get

$$\hat{f}'(x) = \frac{f(x)}{x\log x} \ge \frac{\hat{f}(x)\log x}{C+1+\varepsilon_2}\frac{1}{x\log x} = \frac{\hat{f}(x)}{(C+1+\varepsilon_2)x} \tag{4.11}$$

and

$$\hat{f}'(x) = \frac{f(x)}{x\log x} \le \frac{\hat{f}(x)\log x}{C-\varepsilon_2}\frac{1}{x\log x} = \frac{\hat{f}(x)}{(C-\varepsilon_2)x}. \tag{4.12}$$

Hence, successively, we have

$$\begin{aligned}
\frac{1}{C+1+\varepsilon_2}\frac{1}{x} &\le \frac{\hat{f}'(x)}{\hat{f}(x)} \le \frac{1}{C-\varepsilon_2}\frac{1}{x},\\
\frac{1}{C+1+\varepsilon_3}\log x &\le \log \hat{f}(x) \le \frac{1}{C-\varepsilon_3}\log x,\\
x^{\frac{1}{C+1+\varepsilon_3}} &\le \hat{f}(x) \le x^{\frac{1}{C-\varepsilon_3}}. \qquad (4.13)
\end{aligned}$$

Using once more the inequalities in (4.11) and (4.12), we get, by (4.13), that

$$\frac{1}{C+1+\varepsilon_2}x^{\frac{1}{C+1+\varepsilon_3}} \le \frac{\hat{f}(x)}{C+1+\varepsilon_2} \le \frac{f(x)}{\log x} \le \frac{\hat{f}(x)}{C-\varepsilon_2} \le \frac{1}{C-\varepsilon_2}x^{\frac{1}{C-\varepsilon_3}},$$

which proves Lemma 1.

Lemma 2. *Let f be as in Lemma 1. Then, as $x\to\infty$,*

$$\sum_{p\le x} f(p)\left[\frac{x}{p}\right] = (1+o(1))\int_2^x f(t)\left[\frac{x}{t}\right]\frac{dt}{\log t} + O\left(xf(x)e^{-\sqrt{\log x}}\right). \tag{4.14}$$

Proof. Let $x\ge 2$. We have

$$\begin{aligned}
\sum_{p\le x} f(p)\left[\frac{x}{p}\right] &= \int_{2-0}^x f(t)\left[\frac{x}{t}\right]d\pi(t)\\
&= \int_2^x f(t)\left[\frac{x}{t}\right]\frac{dt}{\log t} + \int_{2-0}^x f(t)\left[\frac{x}{t}\right]dR(t).
\end{aligned}$$

Now

$$\begin{aligned}
\int_2^x f(t)\left[\frac{x}{t}\right]dR(t) &= f(x)R(x) + O(1) - \int_2^x R(t)d\left\{f(t)\left[\frac{x}{t}\right]\right\}\\
&= O\left(xf(x)e^{-\sqrt{\log x}}\right) - \int_2^x R(t)\left[\frac{x}{t}\right]f'(t)\,dt - \int_2^x R(t)f(t)\,d\left[\frac{x}{t}\right].
\end{aligned}$$

We have, using $R(t) \ll te^{-\sqrt{\log t}}$ and integrating by parts,

$$\begin{aligned}
\int_2^x R(t)\left[\frac{x}{t}\right] f'(t)\,dt &\ll x\int_2^x e^{-\sqrt{\log t}} f'(t)\,dt \\
&= x\left(f(x)e^{-\sqrt{\log x}} + O(1) + \frac{1}{2}\int_2^x \frac{f(t)}{\sqrt{\log t}}\frac{e^{-\sqrt{\log t}}}{t}\,dt\right) \\
&\ll xf(x)e^{-\sqrt{\log x}} + \int_2^x \left[\frac{x}{t}\right]\frac{f(t)}{\sqrt{\log t}} e^{-\sqrt{\log t}}\,dt \\
&\ll xf(x)e^{-\sqrt{\log x}} + \int_2^x \left[\frac{x}{t}\right]\frac{f(t)}{\log t}\left(\sqrt{\log t}\; e^{-\sqrt{\log t}}\right)\,dt,
\end{aligned}$$

because $1 \ll f(x)e^{\sqrt{\log x}}$, due to Lemma 1. Thus

$$\int_2^x R(t)\left[\frac{x}{t}\right] f'(t)\,dt = o\left(\int_2^x f(t)\left[\frac{x}{t}\right]\frac{dt}{\log t}\right) + O\left(xf(x)e^{-\sqrt{\log x}}\right).$$

Putting $\frac{x}{t} = u$, we have

$$\begin{aligned}
\int_2^x R(t)f(t)\,d\left[\frac{x}{t}\right] &= \int_{x/2}^1 R\left(\frac{x}{u}\right) f\left(\frac{x}{u}\right) d[u] \\
&= -\sum_{n\le\frac{x}{2}} f\left(\frac{x}{n}\right) R\left(\frac{x}{n}\right) \ll x\sum_{n\le\frac{x}{2}} f\left(\frac{x}{n}\right)\frac{e^{-\sqrt{\log(x/n)}}}{n}.
\end{aligned}$$

Let $0 < \delta < 1$. We use once more Lemma 1 and write

$$\begin{aligned}
\sum_{n\le\frac{x}{2}} f\left(\frac{x}{n}\right)\frac{e^{-\sqrt{\log(x/n)}}}{n} &= \sum_{n\le x^\delta} f\left(\frac{x}{n}\right)\frac{e^{-\sqrt{\log(x/n)}}}{n} + \sum_{x^\delta<n\le\frac{x}{2}} f\left(\frac{x}{n}\right)\frac{e^{-\sqrt{\log(x/n)}}}{n} \\
&\ll f(x)e^{-A(\delta)\sqrt{\log x}}\sum_{n\le x}\frac{1}{n} + f(x^{1-\delta})\sum_{x^\delta<n\le\frac{x}{2}}\frac{1}{n} \\
&\ll f(x)e^{-A_1(\delta)\sqrt{\log x}} + f(x^{1-\delta})\log x \ll f(x)e^{-A_1(\delta)\sqrt{\log x}},
\end{aligned}$$

if δ is a constant sufficiently close to 1. This proves (4.14).

Lemma 3. *Let F be as in Theorem 5. There exist constants $0 < \delta_1 < \delta_2$ and a real number $x_1 > 0$ such that, if $x \ge x_1$, then*

$$x^{\delta_1} \le F(x) \le x^{\delta_2}. \tag{4.15}$$

One can choose $\delta_1 = (C+1)^{-1}$ *and* $\delta_2 = C^{-1}$.

Proof. We have

$$\begin{aligned}(C+o(1))xF(x) &= \int_2^x F(v)\left[\frac{x}{v}\right] dv \\ &= x\int_2^x F(v)\frac{dv}{v} - \int_2^x F(v)\left(\frac{x}{v}-\left[\frac{x}{v}\right]\right) dv = I_1 - I_2,\end{aligned}$$

say. Since $0 \le I_2 \le I_1$, it follows that

$$(C-\varepsilon_1)F(x) \le \int_2^x F(v)\frac{dv}{v} \le (C+1+\varepsilon_1)F(x).$$

This is analogous to (4.10), with $F(x)$ in place of $f(x)/\log x$, and the rest of the proof is as in Lemma 1. Note, however, that we do not need here F to be differentiable, while we needed the differentiability of f in Lemma 1.

Lemma 4. *Let* F, ρ, δ_1 *et* δ_2 *be as above, then*

$$\delta_1 \le \rho \le \delta_2. \tag{4.16}$$

Proof. Since $\delta_1 = \frac{1}{C+1}$ et $\delta_2 = \frac{1}{C}$, we only need to prove that

$$C < \frac{1}{\rho} < C+1.$$

We have

$$C = \int_1^\infty [u]u^{-\rho-2}\,du < \int_1^\infty u^{-\rho-1}\,du = \frac{u^{-\rho}}{-\rho}\Big|_1^\infty = \frac{1}{\rho}.$$

We also have

$$\begin{aligned}C = \int_1^\infty [u]u^{-\rho-2}\,du &> \int_1^\infty (u-1)u^{-\rho-2}\,du = \int_1^\infty u^{-\rho-1}\,du - \int_1^\infty u^{-\rho-2}\,du \\ &= \frac{1}{\rho} - \frac{1}{\rho+1} > \frac{1}{\rho} - 1.\end{aligned}$$

Lemma 5. *The function* $h : \mathbf{R}^+ \to \mathbf{R}^+$ *defined by*

$$h(r) = \frac{\zeta(1+r)}{1+r} \tag{4.17}$$

is strictly decreasing on $\mathbf{R}^+$.

Proof. The result follows from the series representation $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (Re $s > 1$).

Lemma 6. *Let* F *be as in Theorem 5 and let* $\varphi : [2, +\infty[\to \mathbf{R}^+$ *be a function such that* $\lim_{x\to\infty} \varphi(x) = +\infty$ *and* $\varphi(x) = o(x)$. *Then, as* $x \to \infty$,

$$F(x/\varphi(x)) = o(F(x)), \tag{4.18}$$

$$\int_2^{x/\varphi(x)} F(v) \left[\frac{x}{v}\right] dv = o(xF(x)) \tag{4.19}$$

and

$$\int_{x/\varphi(x)}^{x} F(v) \left[\frac{x}{v}\right] dv \sim Cx\, F(x). \tag{4.20}$$

Proof. If x is sufficiently large, we have, appealing to (3.5), that

$$2CxF(x) > \int_{x/\varphi(x)}^{x} F(v) \left[\frac{x}{v}\right] dv \geq \frac{x}{2} F(x/\varphi(x)) \int_{x/\varphi(x)}^{x} \frac{dv}{v} = \frac{x}{2} F\left(\frac{x}{\varphi(x)}\right) \log \varphi(x).$$

It follows from this that

$$F(x/\varphi(x)) \ll \frac{F(x)}{\log \varphi(x)}.$$

Since $\lim_{x\to\infty} \log \varphi(x) = +\infty$, we easily obtain (4.18).

In order to establish (4.19), we proceed as follows: If x is large enough, we have, using (4.18) and (3.5),

$$\begin{aligned} \int_2^{x/\varphi(x)} F(v) \left[\frac{x}{v}\right] dv &= \int_2^{x/\varphi(x)} F(v) \left[\frac{x/\varphi(x)}{v} \cdot \varphi(x)\right] dv \\ &\leq 2\varphi(x) \int_2^{x/\varphi(x)} F(v) \left[\frac{x/\varphi(x)}{v}\right] dv \\ &\ll \varphi(x) \frac{x}{\varphi(x)} F\left(\frac{x}{\varphi(x)}\right) = xF\left(\frac{x}{\varphi(x)}\right) = o(xF(x)), \end{aligned}$$

which proves (4.19). Clearly (4.20) follows from combining (3.5) and (4.19).

Lemma 7. *Let ξ be a real positive number, then*

$$\int_1^x v^\xi \left[\frac{x}{v}\right] dv = x^{1+\xi} h(\xi) + O(x),$$

where $h(r)$ is defined by (4.17).

Proof. Let $\frac{x}{v} = u$, then

$$\begin{aligned}
\int_1^x v^\xi \left[\frac{x}{v}\right] dv &= \int_1^x \left(\frac{x}{u}\right)^\xi [u] \frac{x}{u^2}\, du \\
&= x^{1+\xi} \int_1^x [u] u^{-\xi-2}\, du \\
&= x^{1+\xi} \left(\int_1^\infty [u] u^{-\xi-2}\, du - \int_x^\infty [u] u^{-\xi-2}\, du \right) \\
&= x^{1+\xi} \left(h(\xi) + O\left(\frac{1}{x^\xi}\right)\right),
\end{aligned}$$

whence the result.

5 Proof of Theorem 5 and Theorem 6

Let

$$\Delta_1 \stackrel{\text{def}}{=} \liminf_{x\to\infty} \frac{\log F(x)}{\log x}$$

and

$$\Delta_2 \stackrel{\text{def}}{=} \limsup_{x\to\infty} \frac{\log F(x)}{\log x}.$$

It is clear that these two limits exist (because of Lemma 3) and that

$$0 < \delta_1 \le \Delta_1 \le \Delta_2 \le \delta_2 < +\infty.$$

First, we shall prove that

$$\Delta_1 = \Delta_2. \tag{5.1}$$

Suppose the contrary, i.e. that $\Delta_2 > \Delta_1$. In view of Lemma 4, two cases can occur:

- Case #1. $\rho < \Delta_2$.
- Case #2. $\rho \ge \Delta_2$.

First consider Case #1. Let

$$\xi = \min\left(\frac{\Delta_2 - \Delta_1}{4}, \frac{\Delta_2 - \rho}{2}\right). \tag{5.2}$$

With this choice of ξ, it follows from Lemma 5 that $h(\rho) > h(\Delta_2 - \xi)$. Choose $\varepsilon > 0$ sufficiently small so that

$$\frac{1-\varepsilon}{1+\varepsilon} h(\rho) > h(\Delta_2 - \xi). \tag{5.3}$$

Clearly there exists X_4 such that if $x \ge X_4$, then

$$\log x < x^{\frac{\Delta_2 - \xi}{\Delta_1 + \xi}}. \tag{5.4}$$

Furthermore, because of (4.20) with $\varphi(x) = \log x$, there exists $X_5 = X_5(\varepsilon)$ such that

(5.5)

$$(1-\varepsilon)h(\rho)xF(x) < \int_{x/\log x}^{x} F(v)\left[\frac{x}{v}\right] dv < (1+\varepsilon)h(\rho)xF(x) \quad (x \ge X_5).$$

Set $X = \max(X_4, X_5)$. Then from the definitions of Δ_1 and Δ_2, we have that there exist $y_3 > y_2 > y_1 > X$ such that

$$(\mathcal{I}) \qquad F(y_1) \geq y_1^{\Delta_2-\xi}, \quad F(y_2) \leq y_2^{\Delta_1+\xi}, \quad F(y_3) \geq y_3^{\Delta_2-\xi}.$$

Recall that $F(x)$ is nondecreasing and set

$$\begin{aligned} x_1 &= \max\{x : y_1 < x < y_2 \text{ such that } F(x) = x^{\Delta_2-\xi}\} \\ x_2 &= \max\{x : y_1 < x < y_2 \text{ such that } F(x) = x^{\Delta_1+\xi}\} \\ x_3 &= \max\{x : y_2 < x < y_3 \text{ such that } F(x) = x^{\Delta_2-\xi}\}. \end{aligned}$$

Since $x_1 \leq x_2$ and $F(x)$ is nondecreasing, we have $F(x_1) \leq F(x_2)$ and therefore

$$x_1^{\Delta_2-\xi} \leq x_2^{\Delta_1+\xi},$$

which implies that

$$x_1^{\frac{\Delta_2-\xi}{\Delta_1+\xi}} \leq x_2.$$

From this it follows, taking into account (5.2) and (5.4), that

$$x_1 < \frac{x_2}{\log x_2} < \frac{x_3}{\log x_3} < x_3. \tag{5.6}$$

Using (3.5), (5.5), (5.6) and the fact that $F(v) \leq v^{\Delta_2-\xi}$ for $x_1 \leq v \leq x_3$, we have

$$\begin{aligned} (1-\varepsilon)h(\rho)x_3 x_3^{\Delta_2-\xi} &= (1-\varepsilon)h(\rho)x_3 F(x_3) < \int_{x_3/\log x_3}^{x_3} F(v)\left[\frac{x_3}{v}\right] dv \\ &< \int_{x_1}^{x_3} F(v)\left[\frac{x_3}{v}\right] dv \leq \int_{x_1}^{x_3} v^{\Delta_2-\xi}\left[\frac{x_3}{v}\right] dv \\ &< (1+\varepsilon)h(\Delta_2-\xi)x_3^{1+\Delta_2-\xi}, \end{aligned}$$

which means that

$$(1-\varepsilon)h(\rho) < (1+\varepsilon)h(\Delta_2-\xi),$$

which contradicts (5.3). This proves (5.1) in the case where $\rho < \Delta_2$.

It remains to consider Case #2, that is when $\rho \geq \Delta_2 > \Delta_1$. We proceed essentially as in Case #1. First, we choose

$$\xi = \min\left(\frac{\Delta_2-\Delta_1}{4}, \frac{\rho-\Delta_1}{2}\right).$$

Observing that $\rho > \Delta_1 + \xi$ and that h is decreasing, we choose $\varepsilon > 0$ sufficiently small so that

$$\frac{1+\varepsilon}{1-\varepsilon} h(\rho) < h(\Delta_1 + \xi). \tag{5.7}$$

Appealing again to Lemma 6 (that is, (4.20) with $\varphi(x) = \log x$), we have that there exists $X_6 = X_6(\varepsilon)$ such that

$$(1-\varepsilon)h(\rho)xF(x) < \int_{x/\log x}^{x} F(v) \left[\frac{x}{v}\right] dv < (1+\varepsilon)h(\rho)xF(x) \quad (x \geq X_6). \tag{5.8}$$

Then choose $X = X_6$. From this, we deduce the existence of three numbers $y_3 > y_2 > y_1 > X$ satisfying the inequalities (I) given above. Then define x_1 and x_2 as before.

It follows that, since $F(v) \geq v^{\Delta_1+\xi}$ for $v \in \left[\frac{x_2}{\log x_2}, x_2\right] \subset [x_1, x_2]$, one can write, using (5.8),

$$\begin{aligned} (1+\varepsilon)h(\rho)x_2 x_2^{\Delta_1+\xi} &= (1+\varepsilon)h(\rho)x_2 F(x_2) > \int_{x_2/\log x_2}^{x_2} F(v) \left[\frac{x_2}{v}\right] dv \\ &\geq \int_{x_2/\log x_2}^{x_2} v^{\Delta_1+\xi} \left[\frac{x_2}{v}\right] dv > (1-\varepsilon)h(\Delta_1+\xi)x_2^{1+\Delta_1+\xi}, \end{aligned}$$

which contradicts (5.7). This proves (5.1) in both possible cases. Henceforth let

$$\Delta \stackrel{\text{def}}{=} \Delta_2 = \Delta_1.$$

It remains to prove that

$$\Delta = \rho. \tag{5.9}$$

First observe that, since $\Delta > 0$, relation (3.5) can also be written as

$$\int_1^x F(v) \left[\frac{x}{v}\right] dv \sim CxF(x), \tag{5.10}$$

if we extend the definition of the function F by writing $F(x) = F(2)$ for $x \in [1, 2]$.

In order to prove that $\Delta = \rho$, we proceed by contradiction. Assume first that

$$\Delta > \rho, \quad \text{i.e.} \quad \Delta = \rho + \delta, \quad \delta > 0.$$

Then let

$$F(x) = x^{\Delta} L(x) = x^{\rho+\delta} L(x),$$

with

$$L(x) = x^{o(1)}. \tag{5.11}$$

With this notation, relation (5.10) becomes

$$\int_1^x \frac{L(x/u)}{L(x)} \frac{[u]}{u^{2+\rho+\delta}}\, du \sim C. \tag{5.12}$$

Because of Lemma 5, we have

$$C = \int_1^\infty \frac{[u]}{u^{\rho+2}}\, du = \int_1^x \frac{[u]}{u^{\rho+2}}\, du + O\left(\frac{1}{x^\rho}\right).$$

It follows from this that relation (5.12) can be written as

$$\int_1^x \frac{L(x/u)}{L(x)u^\delta} \frac{[u]}{u^{2+\rho}}\, du \sim \int_1^x \frac{[u]}{u^{2+\rho}}\, du. \tag{5.13}$$

Let now η be a fixed real number satisfying

$$1 < \eta < e^{\delta/2}. \tag{5.14}$$

It also follows from (5.12) that one can choose $X_7 = X_7(\eta)$ such that

$$\frac{\zeta(1+\rho)}{1+\rho} < \sqrt{\eta} \int_1^x \frac{L(x/u)}{L(x)} \frac{[u]}{u^{2+\rho+\delta}}\, du \quad (x \geq X_7). \tag{5.15}$$

Because of relations (5.13) and (5.14), there exists $X_8 = X_8(\eta) > 0$ such that, we have

$$\int_1^x \frac{L(x/u)}{L(x)u^\delta} \frac{[u]}{u^{2+\rho}}\, du \geq \frac{1}{\eta} \int_1^x \frac{[u]}{u^{2+\rho}}\, du \quad (x \geq X_8).$$

It follows from this last inequality that, if we fix a real number x_1 arbitrarily large, $x_1 > X = \max(X_7, X_8)$, then there exists at least one interval

$$[a, b] \subset [1, x_1] \qquad (1 \leq a < b)$$

such that

$$\frac{L(x_1/u)}{L(x_1)u^\delta} \geq \frac{1}{\eta}, \quad \forall u \in [a,b]. \tag{5.16}$$

Amongst these intervals, let $I_1 = [a_{x_1}, b_{x_1}]$ be the one for which the upper bound is the largest.

Now set $v = \frac{x_1}{u}$ in (5.16), so that

$$\frac{L(x_1)}{L(v)} \leq \eta \left(\frac{v}{x_1}\right)^\delta, \quad \forall v \in [x_1/b_{x_1}, x_1/a_{x_1}]. \tag{5.17}$$

Further set $x_2 = x_1/b_{x_1}$. Hence in particular, we shall have

$$\frac{L(x_1)}{L(x_2)} \leq \eta \left(\frac{x_2}{x_1}\right)^\delta. \tag{5.18}$$

If $x_2 \leq X$, then, from (5.18), we have

$$L(x_1) \leq \eta \left(\frac{x_2}{x_1}\right)^\delta L(x_2) \ll x_1^{-\delta},$$

which contradicts our assumption (5.11), since x_1 was choosen arbitrarily large and this ends the proof.

If on the contrary we have $x_2 > X$, then

$$\int_1^{x_2} \frac{L(x_2/u)}{L(x_2)u^\delta} \frac{[u]}{u^{2+\rho}}\, du \geq \frac{1}{\eta} \int_1^{x_2} \frac{[u]}{u^{2+\rho}}\, du. \tag{5.19}$$

As above, we deduce that there exists at least one interval

$$I_2 = [a_{x_2}, b_{x_2}] \subset [1, x_2] \qquad \text{with } 1 \leq a_{x_2} < b_{x_2} \leq b_{x_1}$$

such that

$$\frac{L(x_2/u)}{L(x_2)u^\delta} \geq \frac{1}{\eta}, \quad \forall u \in I_2 \tag{5.20}$$

and that

$$\frac{L(x_2)}{L(v)} \leq \eta \left(\frac{v}{x_2}\right)^\delta, \quad \forall v \in [x_2/b_{x_2}, x_2/a_{x_2}]. \tag{5.21}$$

For each $i = 3, 4, \ldots$, we choose

$$x_i = \frac{x_{i-1}}{b_{x_{i-1}}}$$

and in each case we obtain the existence of two real numbers $1 \le a_{x_i} < b_{x_i} \le b_{x_{i-1}}$ such that

$$\frac{L(x_i)}{L(v)} \le \eta \left(\frac{v}{x_i}\right)^{\delta}, \quad \forall v \in [x_i/b_{x_i}, x_i/a_{x_i}]. \tag{5.22}$$

We continue this process until step n, that is until

$$x_n = x_{n-1}/b_{x_{n-1}} < X < x_{n-1}. \tag{5.23}$$

The fact that there exists a positive integer n for which the above inequalities are satisfied is guaranteed by the fact that

$$b_{x_i} > c_0 > 1, \qquad \forall x_i > X, \tag{5.24}$$

for a certain real number c_0. Indeed, assume for the moment that (5.24) is true. Then (5.23) will certainly hold for some sufficiently large n for which

$$n < \frac{\log(x_1/X)}{\log c_0} + 2. \tag{5.25}$$

Then, using repeatedly (5.22) with $v = x_{i+1} = x_i/b_{x_i}$, we have

$$\begin{aligned}
L(x_1) &= \frac{L(x_1)}{L(x_2)} \cdot \frac{L(x_2)}{L(x_3)} \cdot \frac{L(x_3)}{L(x_4)} \cdot \ldots \cdot \frac{L(x_{n-1})}{L(x_n)} \cdot L(x_n) \\
&\le \eta^{n-1} \cdot \left(\frac{x_2}{x_1}\right)^{\delta} \cdot \left(\frac{x_3}{x_2}\right)^{\delta} \cdot \left(\frac{x_4}{x_3}\right)^{\delta} \cdot \ldots \cdot \left(\frac{x_n}{x_{n-1}}\right)^{\delta} \cdot L(x_n) \\
&= \eta^{n-1} \left(\frac{x_n}{x_1}\right)^{\delta} \cdot L(x_n) < \eta^{n-1} \left(\frac{X}{x_1}\right)^{\delta} \cdot L(x_n) \\
&\ll \frac{\eta^n}{x_1^{\delta}} \ll \frac{\eta^{\log x_1}}{x_1^{\delta}} = \frac{x_1^{\log \eta}}{x_1^{\delta}} < \frac{x_1^{\delta/2}}{x_1^{\delta}} = x_1^{-\delta/2},
\end{aligned}$$

where we used (5.23), (5.14), (5.25) and also the fact that $X^{\delta} L(x_n) \le X^{\delta} \max_{1 \le x \le X} L(x) = O(1)$.

But since x_1 was taken arbitrarily large, this contradicts relation (5.11), which guaranteed that

$$x^{-\varepsilon} \ll L(x) \ll x^{\varepsilon}, \qquad \forall \varepsilon > 0 \text{ fixed} .$$

It remains to prove (5.24), i.e. that we can assume the existence of a constant $c_0 > 1$ such that

$$b_{x_i} > c_0,$$

for all i's for which $x_i > X$ (even if there are infinitely many of them). Assume the contrary. First, if there is only a finite number of b_{x_i}'s for which $x_i > X$, then, since they constitute a monotonic decreasing sequence, they are all bounded below by the smallest amongst them, in which case it is easy to find a $c_0 > 1$ which will act as a lower bound. If on the other hand there exist infinitely many such x_i's with $\lim_{i\to\infty} b_{x_i} = 1$ and with the property

$$\frac{L(x_i/u)}{L(x_i)u^{\delta}} < \frac{1}{\eta}, \quad \forall u \geq b_{x_i}, \tag{5.26}$$

then, let $\varepsilon > 0$ be fixed, arbitrarily small; clearly there exists i_0 such that $b_{x_i} < 1 + \varepsilon$, for all $i \geq i_0$. In particular, since $x_{i_0} > X$, it follows from (5.15) and (5.26), using for short $x = x_{i_0}$,

$$\begin{aligned}
\frac{\zeta(1+\rho)}{1+\rho} &< \sqrt{\eta} \int_1^x \frac{L(x/u)}{L(x)u^{\delta}} \frac{[u]}{u^{2+\rho}} \, du \\
&= \sqrt{\eta} \int_1^{1+\varepsilon} \frac{L(x/u)}{L(x)u^{\delta}} \frac{[u]}{u^{2+\rho}} \, du + \sqrt{\eta} \int_{1+\varepsilon}^x \frac{L(x/u)}{L(x)u^{\delta}} \frac{[u]}{u^{2+\rho}} \, du \\
&< \sqrt{\eta} \int_1^{1+\varepsilon} \frac{L(x/u)}{L(x)u^{\delta}} \frac{[u]}{u^{2+\rho}} \, du + \frac{\sqrt{\eta}}{\eta} \int_{1+\varepsilon}^x \frac{[u]}{u^{2+\rho}} \, du \\
&< \sqrt{\eta} \int_1^{1+\varepsilon} \frac{L(x/u)}{L(x)u^{\delta}} \frac{[u]}{u^{2+\rho}} \, du + \frac{1}{\sqrt{\eta}} \frac{\zeta(1+\rho)}{1+\rho}.
\end{aligned}$$

This last integral tends to 0 as ε tends to 0, which leads us to a contradiction since $\eta > 1$, and this proves (5.24).

This proves that we cannot have $\Delta > \rho$.

In a similar manner one can prove that we cannot have $\Delta < \rho$. Hence (5.9) is true and Theorem 5 is proven.

It remains yet to prove Theorem 6. Assume first that $L(x)$ is nondecreasing for $x \geq X$. Then we have to show

$$\lim_{x\to\infty} \frac{L(x)}{L(ax)} = 1 \tag{5.27}$$

for any $a > 0$. It is easily established that if (5.27) holds for $0 < a < 1$, then it also holds for $a > 1$. Suppose that for some $0 < a < 1$, (5.27) does not hold. Since $L(x) \geq L(ax)$ for sufficiently large x, it follows that there exists a sequence x_i tending to $+\infty$ for which $L(x_i)/L(ax_i) > 1+\eta$ for some $\eta > 0$. Choose $x = x_i$ to be large enough. Then

$$\begin{aligned} L(x)\int_1^\infty h(u)\,du &\sim \int_1^{1/a} L(\frac{x}{u})h(u)\,du + \int_{1/a}^x L(\frac{x}{u})h(u)\,du \\ &\leq L(x)\int_1^{1/a} h(u)\,du + L(ax)\int_{1/a}^x h(u)\,du \\ &\leq L(x)\left(\int_1^{1/a} h(u)\,du + \frac{1}{1+\eta}\int_{1/a}^\infty h(u)\,du\right). \end{aligned}$$

But since $\eta > 0$ this gives

$$\int_1^\infty h(u)\,du \leq \int_1^{1/a} h(u)\,du + \frac{1}{1+\eta}\int_{1/a}^\infty h(u)\,du < \int_1^\infty h(u)\,du,$$

which is a contradiction that proves the theorem when $L(x)$ is nondecreasing. When $L(x)$ is nonincreasing, then the proof is similar. If

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1$$

fails for some $0 < a < 1$, then for some $\eta > 0$ and a sequence x_i tending to $+\infty$, we have $L(ax_i) > (1+\eta)L(x_i)$, since $L(ax) \geq L(x)$ for $x \geq X$. But then, for $x = x_i$,

$$\begin{aligned} L(x)\int_1^x h(u)\,du &\sim L(x)\int_1^\infty h(u)\,du \sim \int_1^x L(\frac{x}{u})h(u)\,du \\ &\geq L(x)\int_1^{1/a} h(u)\,du + L(ax)\int_{1/a}^x h(u)\,du \\ &\geq L(x)\left(\int_1^{1/a} h(u)\,du + (1+\eta)\int_{1/a}^x h(u)\,du\right) \\ &> L(x)\int_1^x h(u)\,du, \end{aligned}$$

and thus again we have a contradiction.

Note, however, that if we drop the hypothesis that $L(x)$ is monotonic, then (3.9) does not necessarily imply that $L \in \mathcal{L}$, which means that the hypothesis of monotonicity is indeed the appropriate one to make. To see this, let $L(x)$ be a piecewise linear function defined as follows: $L(x) = 1$ unless $x \in [2^k - \varepsilon 2^{-k}, 2^k + \varepsilon 2^{-k}]$ ($0 < \varepsilon < \frac{1}{2}$ is a small given number, $k = 1, 2, \ldots$), and $L(2^k) = 2$ for $k = 1, 2, \ldots$. Then obviously

$$\frac{L(x_k)}{L(ax_k)} = 2$$

for $x_k = 2^k$, $k \geq k_0(\varepsilon)$ and say $a = \frac{2}{3}$. Thus $L(x)$ cannot be slowly varying. On the other hand $L(x) = 1 + O(\varepsilon)$ and

$$\begin{aligned}
\int_1^x L(\frac{x}{u})h(u)\,du &= \int_1^x h(u)\,du + O\left(\sum_{k=1}^{O(\log x)} \varepsilon 2^{-k}(x2^{-k})^{-c}\right) \\
&= (1 + O(\varepsilon))\int_1^x h(u)\,du \\
&= \left(1 + O(\varepsilon) + O\left(x^{1-c}\right)\right) L(x)\int_1^\infty h(u)\,du,
\end{aligned}$$

so that (3.9) does hold, since ε may be arbitrarily small.

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