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# ON THE NORMAL GROWTH OF PRIME FACTORS OF INTEGERS

*Dedicated to János Galambos on his 50th birthday*

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ABSTRACT. Let $h: [0,1] \longrightarrow \mathbf{R}$ be such that $\int_0^1 \frac{|h(u)|}{u}\,du < +\infty$ and define $T_h(n,y) = T(n,y) = \sum_{q|n,\, q<y} h\Big(\frac{\log q}{\log y}\Big)$. In 1966, Erdős [8] proved that

$$\max_{p|n} \frac{1}{\log p} \sum_{\substack{q^\alpha \| n \\ q<p}} \alpha \log q = \Big(1 + o(1)\Big) \frac{\log\log\log n}{\log\log\log\log n}$$

holds for almost all $n$, which by using a simple argument implies that in the case $h(u) = u$, for almost all $n$,

$$\max_{p|n} T(n,p) = \Big(1 + o(1)\Big) \frac{\log\log\log n}{\log\log\log\log n}.$$

He further obtained that, for every $z > 0$ and almost all $n$,

$$\frac{1}{\log\log n} \#\{p|n : T(n,p) < z\} = \Big(1 + o(1)\Big)\varphi(z)$$

and that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : (\log\log n) \min_{p|n} T(n,p) < z\} = \psi(z),$$

where $\varphi, \psi$ are continuous distribution functions. Several other results concerning the normal growth of prime factors of integers were obtained by Galambos [10], [11] and by De Koninck and Galambos [6].

Let $\chi = \{x_m : m \in \mathbf{N}\}$ be a sequence of real numbers such that $\lim_{m\to\infty} x_m = +\infty$. For each $x \in \chi$ let $\wp_x$ be a set of primes $p \le x$. Denote by $p(n)$ the smallest prime factor of $n$. In this paper, we investigate the number of prime divisors $p$ of $n$, belonging to $\wp_x$, for which $T_h(n,p) < z$. Given $\Delta > 1$, we study the behaviour of the function $k(n) = \max_{p|n,\, p\in\wp_x} \#\{q|n : p^{1/\Delta} < q < p\}$. We also investigate the two functions $k^*(n) = \max_{p|n,\, p\in\wp_x} T_h(n,p)$ and $\Upsilon(n) = \min_{p|n,\, p\in\wp_x,\, p>p(n)} T_h(n,p)$, where, in each case, $h$ belongs to a large class of functions.

1. **Introduction.** For an integer $n \ge 2$, we denote by $P(n)$ its largest prime factor and by $p(n)$ its smallest prime factor. The letters $p, q, P, Q$ stand for prime numbers. For a real number $y \ge 1$, let

$$n_y \stackrel{\text{def}}{=} \prod_{p^\alpha \| n;\; p<y} p^\alpha,$$

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an empty product being counted as 1. By $\nu_x\{n \le x : \cdots\}$, we mean the frequency of the integers $1 \le n \le x$ for which the property stated in the dotted space holds.

Given an integer $n \ge 2$, let $p_1 < p_2 < \cdots < p_\omega$, $\omega = \omega(n)$, be its distinct prime divisors, that is, $p_j = p_j(n)$. Galambos [10] proved that, for $z > 1$,

$$\lim_{x\to\infty} \nu_x\left\{n \le x : \frac{\log p_{j+1}(n)}{\log p_j(n)} < z\right\} = 1 - \frac{1}{z},$$

if $j = j(x)$ is a function which goes to $+\infty$ as $x \longrightarrow \infty$ but also satisfies "$j(x) \le (1-\varepsilon)\log\log x$' for some $\varepsilon > 0$.

In [11], Galambos proved that, if, as $x \longrightarrow \infty$, both $y = y(x)$ and $\frac{\log x}{\log y(x)}$ tend to $+\infty$, then

$$\lim_{x\to\infty} \nu_x\left\{n \le x : \frac{\log P(n_y)}{\log y} < u,\ \frac{\log P\big((n+1)_y\big)}{\log y} < v\right\} = uv$$

for $0 \le u \le 1$, $0 \le v \le 1$. He concluded from this that, denoting by $p(n,x,y)$ the largest prime divisor of $n$ that does not exceed $y$ (with $y = y(x)$ as above), the natural density of those $n \le x$ for which $p(n,x,y) < p(n+1,x,y)$ equals $\frac{1}{2}$.

In 1987, J. M. De Koninck and J. Galambos [6] proved that $\log\log p_j$ forms a limiting Poisson process if $j$ goes through the indices for which $p_j$ is an intermediate prime divisor. More precisely, they proved that, if $j = j(x)$ is a function which goes to $+\infty$ as $x \longrightarrow \infty$ and if both $\lim_{n\to\infty} p_j(n) = +\infty$ and $\lim_{x\to\infty} \frac{\log p_j(n)}{\log x} = 0$ (where $1 \le n \le x$), then the points $\log\log p_{j+k}$, $k \ge 1$, form a Poisson process in limit as $x \longrightarrow \infty$.

In 1946, Erdős[7] considered the sequence $\eta_i = \frac{\log p_{i+1}}{\log p_i}$ ($i = 1, 2, \ldots, \omega - 1$) and proved that, for almost all $n$, the number of $\eta_i$'s not exceeding $t$ ($t > 1$) is $\big(1 + o(1)\big)$ $(1 - \frac{1}{t})\log\log n$. In 1950, he investigated [8] the sequence $\frac{\log n_{p_i}}{\log p_i}$ (see (1.3) below).

Let us now consider a more general setup. Given a function $h$: $[0, 1) \longrightarrow \mathbf{R}$, if $n < x$, let

$$u_x(n) \stackrel{\text{def}}{=} \sum_{p|n} h\left(\frac{\log p}{\log x}\right); \quad v(n) \stackrel{\text{def}}{=} \sum_{p|n} h\left(\frac{\log p}{\log P(n)}\right). \tag{1.1}$$

We shall assume that

$$\int_0^1 \frac{|h(u)|}{u}\, du < +\infty.$$

For the sake of clarity and simplicity, especially in the statement of the theorems and their proofs, we shall assume that the domain of $h$ is extended to $[0, \infty)$ and that $h(u) = 0$ for $u \ge 1$.

In [4], we proved that, in the case $h(u) = u^\alpha$ with $\alpha > 0$, $u_x(n)$ and $v(n)$ have limit distributions. One can easily see that under quite general conditions on $h$, the functions $u_x(n)$ and $v(n)$ will still both have limit distributions. In [5], we investigated the continuity module of the limit distribution in the case $h(u) = u^\alpha$, $\alpha > 0$.

Let

$$T_h(n, y) = T(n, y) \stackrel{\text{def}}{=} \sum_{q|n_y} h\left(\frac{\log q}{\log y}\right). \tag{1.2}$$

In 1966, Erdős [8] proved that, for almost all $n$,

$$\max_{p|n} \frac{1}{\log p} \sum_{\substack{q^\alpha \| n \\ q<p}} \alpha \log q = \big(1 + o(1)\big) \frac{\log\log\log n}{\log\log\log\log n},$$

which by using a simple argument implies that if $h(u) = u$, then, for almost all $n$,

$$\max_{p|n} T(n,p) = \big(1 + o(1)\big) \frac{\log\log\log n}{\log\log\log\log n}. \tag{1.3}$$

He further obtained that, for every $z > 0$ and almost all $n$,

$$\frac{1}{\log\log n} \#\{p \mid n : T(n,p) < z\} = \big(1 + o(1)\big)\varphi(z) \tag{1.4}$$

and that

$$\lim_{x\to\infty} \nu_x\{n \le x : (\log\log n) \min_{p|n} T(n,p) < z\} = \psi(z), \tag{1.5}$$

where $\varphi, \psi$ are continuous distribution functions.

In [1], J. D. Bovey sharpened (1.3) and (1.4) and determined $\varphi$.

In this paper, we consider estimates similar to those of (1.3)–(1.5) but for the more general function $T_h(n, y)$.

In Section 2, we establish the necessary tools.

Let $\chi = \{x_m : m \in \mathbf{N}\}$ be a sequence of real numbers such that $\lim_{m\to\infty} x_m = +\infty$. For each $x \in \chi$ let $\wp_x$ be a set of primes $p \le x$. In Section 3, we study the number of prime divisors $p$ of $n$, belonging to $\wp_x$, for which $T_h(n,p) < z$. In Section 4, we study the function $k(n) = \max_{p|n, p\in\wp_x} \alpha(n,p)$, where $\alpha(n,y)$ stands for the number of distinct prime divisors $q$ of $n$ which are located in the interval $(y^{1/\Delta}, y)$, for a preassigned $\Delta > 1$. In Section 5, we investigate the function $k^*(n) = \max_{p|n,p\in\wp_x} T_h(n,p)$ for a particular function $h$. In Section 6, we analyze some of the distribution functions connected with the distribution of the prime divisors. Finally in Section 7, we are interested in a problem analogous to the estimate (1.5) of Erdős, namely that of estimating $\Upsilon(n) = \min_{p|n,p\in\wp_x,p>p(n)} T_h(n,p)$.

Throughout the text, we shall use the notion of *weak convergence*. A sequence $F_n(x)$ of distribution functions is said to *converge weakly* to the distribution function $F(x)$ if $F_n(x) \longrightarrow F(x)$ at each continuity point $x$ of $F(x)$ as $n \longrightarrow \infty$. If, in addition, $F_n(-\infty) \longrightarrow F(-\infty)$ and $F_n(+\infty) \longrightarrow F(+\infty)$ we say that $F_n(x)$ converges to $F(x)$ *completely*.

2. **Preliminary results.** Let $\Psi(x,y) = \#\{n \le x : P(n) \le y\}$ and $\Phi(x,y) = \#\{n \le x : p(n) > y\}$. It is known (see de Bruijn [2], [3]) that

$$\Psi(x,y) < x \exp\left(-c\frac{\log x}{\log y}\right) \tag{2.1}$$

and

$$\Phi(x,y) = x \prod_{q\le y}\left(1 - \frac{1}{q}\right)\left(1 + O\left(e^{-a\frac{\log x}{\log y}}\right)\right) \tag{2.2}$$

uniformly for $2 \le y \le x$, where $a, c$ are positive absolute constants.

LEMMA 1. *Let $f$ be a strongly multiplicative function such that $|f(n)| \leq 1$ and $f(p) = 1$ for every prime $p > y$. Then, for $2 \leq y \leq x$,*

$$\frac{1}{x} \sum_{n \leq x} f(n) = \prod_{q \leq y} \left( 1 + \frac{f(q) - 1}{q} \right) + O\left(e^{-c_1 \frac{\log x}{\log y}}\right). \tag{2.3}$$

*Furthermore, if $D$ is a square free integer such that $P(D) \leq y$, then*

$$\sum_{n \leq x,\ n \equiv 0 \ (\mathrm{mod}\ D)} f(n) = x \frac{f(D)}{D} \prod_{q \leq y; q \nmid D} \left( 1 + \frac{f(q) - 1}{q} \right) + O\left( x \frac{e^{-c_1 \frac{\log x/D}{\log y}}}{\varphi(D)} \right), \tag{2.4}$$

*The constants implied by the $O$ terms are absolute and $c_1 = \min(a, \frac{c}{2})$.*

PROOF. We shall only prove (2.3), since (2.4) is an immediate consequence of it. For this, write each positive integer $n \leq x$ as $n = n_1 n_2$, where $P(n_1) \leq y$ and $p(n_2) > y$ so that $f(n) = f(n_1) f(n_2) = f(n_1)$. Then we have

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n_1 \leq x} f(n_1) \sum_{n_2 \leq x/n_1} 1 = \sum_{n_1 \leq x} f(n_1) \Phi\left( \frac{x}{n_1}, y \right) \\ &= x \sum_{n_1 \leq x} \frac{f(n_1)}{n_1} \prod_{q \leq y} \left( 1 - \frac{1}{q} \right) + O\left(x e^{-a \frac{\log x}{\log y}}\right) \\ &= x \sum_{n_1 = 1}^{\infty} \frac{f(n_1)}{n_1} \prod_{q \leq y} \left( 1 - \frac{1}{q} \right) + O\left( \frac{x}{\log y} \sum_{n_1 > x} \frac{1}{n_1} \right) + O\left(x e^{-a \frac{\log x}{\log y}}\right). \end{aligned} \tag{2.5}$$

But

$$\begin{aligned} \sum_{n_1 > \sqrt{x}} \frac{1}{n_1} &\ll \int_{\sqrt{x}}^{\infty} \frac{1}{t} \, d\Psi(t, y) \\ &= \frac{1}{t} \Psi(t, y) \Big|_{\sqrt{x}}^{\infty} + \int_{\sqrt{x}}^{\infty} \frac{\Psi(t, y)}{t^2} \, dt \\ &\ll e^{-\frac{c}{2} \frac{\log x}{\log y}} + \int_{\sqrt{x}}^{\infty} e^{-c \frac{\log t}{\log y}} \frac{dt}{t} \ll \log y \, e^{-\frac{c}{2} \frac{\log x}{\log y}}. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), then (2.3) follows immediately.

LEMMA 2 [TURAN-KUBILIUS INEQUALITY]. *Let $f$ be a complex valued strongly additive function and set*

$$a(x) = \sum_{p \leq x} \frac{f(p)}{p}, \quad b(x) = \sum_{p \leq x} \frac{|f(p)|^2}{p}.$$

*Then*

$$\sum_{n \leq x} |f(n) - a(x)|^2 \leq c x b(x).$$

*For the proof, see Kubilius [16].*

As an immediate consequence of Lemma 2, one can deduce a well known theorem of Hardy and Ramanujan [14], namely that, for almost all positive integers $n$,

$$\omega(n) = \big(1 + o(1)\big) \log \log n.$$

LEMMA 3. *Let h be a Riemann integrable bounded function in* $[0, 1]$, *monotonic in a neighbourhood of* $0$, *furthermore assume that both* $\lim_{u\to 0} h(u) = 0$ *and* $\int_0^1 \frac{|h(u)|}{u}\,du < +\infty$ *hold; finally, set*

$$\varphi_y(\tau) \stackrel{\text{def}}{=} \prod_{q<y}\left(1 + \frac{e^{i\tau h(\frac{\log q}{\log y})} - 1}{q}\right). \tag{2.7}$$

*Then*

$$\lim_{y\to\infty} \varphi_y(\tau) = \exp\left\{\int_0^1 \frac{e^{i\tau h(v)} - 1}{v}\,dv\right\} \stackrel{\text{def}}{=} \exp\{\alpha(\tau)\} \stackrel{\text{def}}{=} \varphi(\tau) \tag{2.8}$$

*and the convergence is uniform for* $\tau$ *varying in a bounded interval.*

PROOF. As we will see, the proof is essentially an easy consequence of the Prime Number Theorem. Let $|\tau| \le c$. If $y$ is large, then

$$\left|1 + \frac{e^{i\tau h(\frac{\log q}{\log y})} - 1}{q}\right| \ge \frac{1}{3},$$

and so

$$|\varphi_y(\tau)| \ge \frac{1}{3}\prod_{3\le q\le y}\left(1 - \frac{1}{q}\right).$$

Let $\delta_n$ and $\varepsilon_n$ be two sequences of positive numbers such that $\lim_{n\to\infty}\delta_n = 0$ and that $\lim_{n\to\infty}\varepsilon_n \log(1/\delta_n) = 0$. Further define $h_n(x)$ as a step function such that both

$$\max_{\delta_n\le x\le 1} |h_n(x) - h(x)| \le \varepsilon_n, \text{ and } h_n(x) = 0 \text{ for } x \in [0, \delta_n]$$

hold. Then, by using elementary estimates on the distribution of primes, we get that

$$\limsup_{y\to\infty} \sum_{q<y} \frac{\left|e^{i\tau h(\frac{\log q}{\log y})} - e^{i\tau h_n(\frac{\log q}{\log y})}\right|}{q} \le c_1 \int_0^{\delta_n} \frac{|h(u)|}{u}\,du + c_2\varepsilon_n \log\frac{1}{\delta_n}.$$

From the Prime Number Theorem it is clear that

$$\lim_{y\to\infty} \sum_{q<y} \frac{e^{i\tau h_n(\frac{\log q}{\log y})} - 1}{q} = \int_0^1 \frac{e^{i\tau h_n(u)} - 1}{u}\,du.$$

But this last integral tends to $\alpha(\tau)$ as $n \to \infty$. Hence to finish the proof it is enough to observe that

$$\limsup_{y\to\infty}\left|\log\varphi_y(\tau) - \sum_{q<y}\frac{e^{i\tau h_n(\frac{\log q}{\log y})} - 1}{q}\right|$$
$$\le \limsup_{y\to\infty}\sum_{q<y}\frac{\left|e^{i\tau h(\frac{\log q}{\log y})} - 1\right|^2}{q^2} + c_1\int_0^{\delta_n}\frac{|h(u)|}{u}\,du + c_2\varepsilon_n\log\frac{1}{\delta_n},$$

which clearly tends to 0 as $n \to \infty$. Therefore $\lim_{y\to\infty}\log\varphi_y(\tau) = \alpha(\tau)$, which means that $\lim_{y\to\infty}\varphi_y(\tau) = \varphi(\tau)$.

EXAMPLES.

1. If $(0 <)\ a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k\ (\leq 1)$ and

$$h(u) = \begin{cases} 1 & \text{if } u \in \cup[a_j, b_j), \\ 0 & \text{otherwise}, \end{cases}$$

then

$$\alpha(\tau) = (e^{i\tau} - 1)\sum_{i=1}^{k} \log \frac{b_i}{a_i}.$$

2. If $h(v) = v^\beta$, $\beta > 0$, then

$$\alpha(\tau) = \frac{1}{\beta}\int_0^\tau \frac{e^{iv} - 1}{v}\, dv.$$

3. If $h(v) = (1 + \log \frac{1}{v})^{-\gamma}$, $\gamma > 1$, then

$$\alpha(\tau) = \frac{\tau^{1/\gamma}}{\gamma}\int_0^\tau (e^{iz} - 1) z^{-1-1/\gamma}\, dz.$$

REMARK. Professor László Szeidl kindly informed us that the following assertions are true:

1. If $h$ is monotonic, then the distribution function $F$, the characteristic function of which is $\varphi(\tau)$, is infinitely divisible. His proof goes as follows. According to a classical theorem due to Gnedenko, $F$ is infinitely divisible if its characteristic function $\varphi(\tau) = e^{\alpha(\tau)}$ has the form

$$(*) \qquad \alpha(\tau) = i\gamma\tau - \frac{\sigma^2\tau^2}{2} + \int_{\substack{-\infty \\ x \neq 0}}^{\infty} \left(e^{i\tau x} - 1 - \frac{i\tau x}{1 + x^2}\right) dL(x),$$

(for the validity of $(*)$, see Galambos [12], pp. 191, 195), where $L(-\infty) = L(+\infty) = 0$, $L$ is nondecreasing on the semi-axis $x < 0$ and $x > 0$, and

$$(**) \qquad \int_{0<|x|<1} x^2\, dL(x) < +\infty$$

holds. From this it follows that

$$\alpha(\tau) = \int_0^1 (e^{i\tau h(v)} - 1)\frac{dv}{v} = \int (e^{i\tau h(v)} - 1)\, d\log v = \int (e^{i\tau u} - 1)\, d\log\big(h^{-1}(u)\big),$$

where $h^{-1}(u)$ denotes the inverse function of $h$. Letting $L(u) = \log h^{-1}(u)$, we have

$$\int u^2\, dL(u) = \int h^2(v)\, d\log v = \int \frac{h^2(v)}{v}\, dv < +\infty.$$

Hence it is clear that $\alpha(\tau)$ can be written in the form $(*)$ and that $(**)$ is satisfied.

2. Assume moreover that $\log h^{-1}(u)$ is absolutely continuous and that $F$ has a finite expectation. Then $F$ has a density function $f$, and $f$ is the solution of the integral equation
$$xf(x) = \int_{y\neq 0} f(x-y)y\, d\big(\log h^{-1}(y)\big).$$
This is an immediate consequence of a theorem due to V. M. Zolotarev (see [19], Lemma 2.7.6, p. 134).

Let $F(z)$ denote the distribution function that corresponds to $\exp\{\alpha(\tau)\}$.

THEOREM 1. *Under the conditions stated in Lemma 3, if* $y = y(x) \longrightarrow \infty$ *and* $\frac{\log x}{\log y(x)} \longrightarrow \infty$, *as* $x \longrightarrow \infty$, *then*
$$\lim_{x\longrightarrow\infty} \nu_x\{n \le x : T(n,y) < z\} = F(z)$$
*completely.*

PROOF. Let
$$f(q) \overset{\text{def}}{=} e^{i\tau h(\frac{\log q}{\log y})}$$
and substitute it in Lemma 1, then, using Lemma 3, it follows that
$$\frac{1}{x}\sum_{n\le x} e^{i\tau T(n,y)} = \varphi_y(\tau) + O\Big(e^{-a\frac{\log x}{\log y}}\Big),$$
which converges to $\varphi(\tau)$ if $y = y(x) \longrightarrow \infty$ and satisfies the condition of the theorem.

LEMMA 4. *Let r be a positive integer. Further let* $1 < y_1(x) < y_2(x) < \cdots < y_r(x) < y_{r+1}(x) = x$ *and* $r(x)$ *be functions of x for which*
$$r(x) \longrightarrow \infty, \quad \log y_1(x) \ge r(x), \quad \frac{\log y_{j+1}(x)}{\log y_j(x)} \ge r(x) \quad (j = 1, 2, \dots, r)$$
*hold. Assume that h satisfies the conditions stated in Lemma 3. Let* $\tau_1, \tau_2, \dots, \tau_r$ *be located in a bounded interval,* $\max_j |\tau_j| \le B$. *Further set*
$$\sigma_q \overset{\text{def}}{=} \sum_{j=1}^{r} \tau_j h\left(\frac{\log q}{\log y_j}\right)$$
*and*
$$\sigma_x(\tau_1, \dots, \tau_r) = \prod_{q\le y_r}\left(1 + \frac{e^{i\sigma_q} - 1}{q}\right). \tag{2.7}$$
*Then, for every large* $x \ge x_0(B)$, *we have*
$$\left|\frac{\sigma_x(\tau_1, \dots, \tau_r)}{\varphi(\tau_1)\dots\varphi(\tau_r)} - 1\right| \le \rho\big(r(x), B\big),$$
*where* $\rho(u, B) \longrightarrow 0$ *monotonically as* $u \longrightarrow \infty$.

PROOF. The proof is similar to the one of Lemma 3. Let $y_0 = y_0(x)$ be defined by $\log y_0(x) = \frac{\log y_1(x)}{r(x)}$. We write (2.7) as $\prod^{(0)} \cdots \prod^{(r)}$ where in $\prod^{(0)}$, the product runs over those $q \le y_0$, and in $\prod^{(j)}$, the product runs over those $q \in (y_{j-1}, y_j]$. Clearly we have

$$\log\left|\prod^{(0)}\right| \ll \sum_{q\le y_0} \frac{|e^{i\sigma_q}-1|}{q} \ll B \sum_{q\le y_0} \frac{1}{q} \sum_{j\le r}\left|h\left(\frac{\log q}{\log y_j}\right)\right|,$$

which is $\ll \int_0^{1/r(x)} \frac{|h(u)|}{u}\, du$. Similarly one can see that

$$\log\left|\frac{\varphi_{y_j}(\tau)}{R_j}\right| \ll \int_0^{1/r(x)} \frac{|h(u)|}{u}\, du,$$

where

$$R_j(\tau) = \prod_{y_{j-1}<q\le y_j}\left(1 + \frac{e^{i\tau h(\frac{\log q}{\log y_j})} - 1}{q}\right).$$

But we also have

$$\log \frac{\prod^{(j)}}{R_j(\tau_j)} = \sum_{y_{j-1}<q\le y_j} \frac{e^{i\sigma_q} - e^{i\tau_j h(\frac{\log q}{\log y_j})}}{q} + O\left(\sum \frac{1}{q^2}\right).$$

The main sum above is smaller than

$$\sum_{q\le y_j} \frac{\left|\sigma_q - \tau_j h\left(\frac{\log q}{\log y_j}\right)\right|}{q} \ll \sum_{\ell=j+1}^{r} \sum_{q\le y_j} \frac{\left|h\left(\frac{\log q}{\log y_\ell}\right)\right|}{q} \ll \int_0^{1/r(x)} \frac{|h(u)|}{u}\, du.$$

Combining the above estimates, we immediately obtain Lemma 4.

As an immediate consequence of this lemma, we mention the following:

THEOREM 2. *Under the conditions stated in Lemma 4, one has*

$$\lim_{x\to\infty} \nu_x\{n \le x : T(n, y_j) < z_j,\ j = 1, 2, \ldots, r\} = F(z_1)\ldots F(z_r)$$

*completely.*

We now state a refinement of the Berry Esseen Inequality due to Fainleib [9] and which can be found in the book of A. G. Postnikov ([17]; Section 1.4, Theorem and Corollary 1).

LEMMA 5. *Suppose that $F(x)$ and $G(x)$ are distribution functions and that $f(t)$ and $g(t)$ are their corresponding characteristic functions. Then, for $T > 0$,*

$$\sup_x |F(x) - G(x)| < c_1\left(S_G(1/T) + \int_0^T |f(t) - g(t)|\frac{dt}{t}\right),$$

*where $c_1$ is an absolute constant and*

$$S_G(h) = \sup_x \frac{1}{2h}\int_0^h \big(G(x+u) - G(x-u)\big)\, du. \tag{2.8}$$

*Moreover, if we let*

$$Q_G(h) \stackrel{\text{def}}{=} \sup_{-\infty<x<+\infty} \big(G(x+h) - G(x)\big),$$

*then*

$$Q_G(h) \le c_2 \sup_{t\ge 1/h} \frac{1}{t}\int_0^t |g(u)|\, du.$$

3. **Sampling the function $T(n,p)$ at some prime divisors $p$ of $n$.** Let $\chi = \{x_m : m \in \mathbf{N}\}$ be a sequence of real numbers such that $\lim_{m\to\infty} x_m = +\infty$. For each $x \in \chi$ let $\wp_x$ be a set of primes $p \le x$. Set

$$\xi(\wp_x) \stackrel{\text{def}}{=} \sum_{p\in\wp_x} \frac{1}{p} \tag{3.1}$$

and

$$\omega_{\wp_x}(n) \stackrel{\text{def}}{=} \#\{p|n : p \in \wp_x\}.$$

Recall that

$$T_h(n,y) = T(n,y) = \sum_{q|n_y} h\left(\frac{\log q}{\log y}\right).$$

THEOREM 3. *Let*

$$s(n;z) \stackrel{\text{def}}{=} \frac{1}{\omega_{\wp_x}(n)} \#\{p|n :\ p \in \wp_x,\ T(n,p) < z\}. \tag{3.2}$$

*Assume that $\xi(\wp_x) \longrightarrow \infty$ and that $h$ satisfies the conditions stated in Lemma 3. Then,*

$$\lim_{x\to\infty,\ x\in X} \frac{1}{x}\sum_{n\le x} |s(n,z) - F(z)| = 0$$

*at each continuity point $z$ of $F(z)$, and at $z = -\infty$ and $z = +\infty$.* (Recall that $F(z)$ is the distribution function that corresponds to $\varphi(t) = \exp\big(\alpha(t)\big)$).

PROOF. Let

$$A(n,\tau) = \sum_{p|n,\, p\in\wp_x} e^{i\tau T(n,p)}.$$

Then $A(n,\tau)/\omega_{\wp_x}(n)$ is the characteristic function of $s(n,z)$. Because of the continuity theorem of characteristic functions, it is enough to prove that

$$\sup_{|\tau|\le B} \frac{1}{x}\sum_{n\le x}\left|\frac{A(n,\tau)}{\omega_{\wp_x}(n)} - \varphi(\tau)\right| \longrightarrow 0 \text{ as } x \longrightarrow \infty. \tag{3.3}$$

(If $\omega_{\wp_x}(n) = 0$, we set $\frac{A(n,\tau)}{\omega_{\wp_x}(n)} = 0$.)

First observe that $\left|\frac{A(n,\tau)}{\omega_{\wp_x}(n)}\right| \le 1$. Since Lemma 2 implies

$$\sum_{n\le x} |\omega_{\wp_x}(n) - \xi(\wp_x)|^2 \le Cx\xi(\wp_x),$$

it follows immediately that

$$\frac{1}{x}\#\{n \le x : |\omega_{\wp_x}(n) - \xi(\wp_x)| > \xi(\wp_x)^{3/4}\} \le \frac{C}{\sqrt{\xi(\wp_x)}} \longrightarrow 0 \text{ as } x \longrightarrow \infty.$$

Thus the contribution in (3.3) of the integers $n \le x$ for which $|\omega_{\wp_x}(n) - \xi(\wp_x)| > \xi(\wp_x)^{3/4}$ is $o(1)$. So assuming that $|\omega_{\wp_x}(n) - \xi(\wp_x)| \le \xi(\wp_x)^{3/4}$, it follows that

$$\left|\frac{A(n,\tau)}{\omega_{\wp_x}(n)} - \frac{A(n,\tau)}{\xi(\wp_x)}\right| \le \frac{|A(n,\tau)|\,|\omega_{\wp_x}(n) - \xi(\wp_x)|}{\omega_{\wp_x}(n)\xi(\wp_x)} \le \xi(\wp_x)^{-1/4}.$$

Thus it is enough to prove that

$$\sup_{|\tau|\le B} \frac{1}{x}\sum_{n\le x}\left|\frac{A(n,\tau)}{\xi(\wp_x)} - \varphi(\tau)\right| \longrightarrow 0 \quad \text{as } x \longrightarrow \infty. \tag{3.4}$$

Let $\varepsilon(x)$ be a function defined on $\mathcal{X}$ such that $\lim_{x\to\infty}\varepsilon(x) = 0$ and

$$\frac{1}{\varepsilon(x)} = o\big(\xi(\wp_x)\big) \tag{3.5}$$

holds. Let $u(x)$ and $v(x)$ be defined by the relations

$$\log\log u(x) = \varepsilon(x)\xi(\wp_x), \tag{3.6}$$

$$\log\frac{\log x}{\log v(x)} = \varepsilon(x)\xi(\wp_x). \tag{3.7}$$

Therefore $u(x) \longrightarrow \infty$ and $v(x) = x^{o(1)}$. Further define

$$J_1 = [u(x), v(x)],$$
$$J_2 = [1,x] \setminus J_1,$$
$$\omega_j(n) = \#\{p : p \mid n, p \in \wp_x, p \in J_j\} \quad (j = 1,2),$$
$$\xi_j(\wp_x) = \sum_{p\in\wp_x,\, p\in J_j} \frac{1}{p}.$$

Since each prime $p \in J_2$ satisfies one of the two inequalities "$p < u(x)$" or "$v(x) < p \le x$", it follows that $\xi_2(\wp_x) < 3\varepsilon(x)\xi(\wp_x)$. Also set

$$A_1(n,\tau) \stackrel{\text{def}}{=} \sum_{p|n,\, p\in J_1,\, p\in\wp_x} e^{i\tau T(n,p)}, \quad c(n,\tau) \stackrel{\text{def}}{=} \frac{A_1(n,\tau)}{\xi(\wp_x)\varphi(\tau)}.$$

Clearly we have

$$|A(n,\tau) - A_1(n,\tau)| \le \omega_2(n) \text{ and } \sum_{n\le x}\omega_2(n) \ll x\varepsilon(x)\xi(\wp_x).$$

Moreover it follows from the Turan-Kubilius Inequality that the normal order of $\omega_1(n)$ is $\xi_1(\wp_x)$. Hence, setting

$$D_x(\tau) \stackrel{\text{def}}{=} \sum_{n\le x} |c(n,\tau) - 1|^2, \tag{3.8}$$

it follows that, if we can prove that

$$\lim_{x\to\infty} \frac{D_x(\tau)}{x} = 0, \tag{3.9}$$

then (3.4) will be proven. Indeed

$$\begin{aligned}\sum_{n\le x}\left|\frac{A(n,\tau)}{\xi(\wp_x)} - \varphi(\tau)\right| &= \sum_{n\le x} |\varphi(t)| \left|\frac{A(n,\tau)}{\xi(\wp_x)\varphi(\tau)} - 1\right| \\ &\le \sum_{n\le x} |c(n,\tau) - 1| + \sum_{n\le x} \frac{|A(n,\tau) - A_1(n,\tau)|}{\xi(\wp_x)} = \Sigma_1 + \Sigma_2.\end{aligned}$$

Then clearly

$$\Sigma_2 \ll \frac{1}{\xi(\wp_x)} \sum_{n\le x} \omega_2(n) \ll x\varepsilon(x),$$

and furthermore, by the Cauchy-Schwarz inequality,

$$\Sigma_1 \ll \sqrt{x}\sqrt{D_x(\tau)} = o(x).$$

To prove (3.9), we proceed as follows. Define $E_1 = \sum_{n\le x} |c(n,\tau)|^2$, $E_2 = \sum_{n\le x} c(n,\tau)$ so that

$$D_x(\tau) = E_1 - 2\Re(E_2) + [x]. \tag{3.10}$$

We first estimate $E_2$. We observe that

$$\sum_{n\le x} A_1(n,\tau) = \sum_{p\in J_1} \sum_{n\equiv 0 \,(\mathrm{mod}\, p)} e^{i\tau T(n,p)} = \sum_{p\in J_1} S_p,$$

say. We now set $f(n) = f_p(n) = e^{i\tau T(n,p)}$ in Lemma 1; note that for such a prime $p \in J_1$, one has $\frac{\log x}{\log p} > e^{\varepsilon(x)\xi(\wp_x)} \stackrel{\text{def}}{=} \rho_1(x)$ (with $\rho_1(x) \to \infty$ as $x \to \infty$). Hence, applying Lemma 1, we get that

$$S_p = \frac{x}{p}\varphi_p(\tau) + O\Big(\frac{x}{p}\exp\big(-c_1\rho_1(x)\big)\Big)$$

uniformly for $p \in J_1$.

It follows from this that

$$E_2 = \frac{x}{\xi(\wp_x)} \sum_{p\in J_1} \frac{\varphi_p(\tau)}{p\varphi(\tau)} + O\Big(\frac{x}{|\varphi(\tau)|}\exp\big(-c_1\rho_1(x)\big)\Big). \tag{3.11}$$

Clearly $\varphi(\tau)$ is never zero. From now on we assume that $\tau$ is bounded, say $|\tau| \leq B$. It follows from Lemma 3 that $\varphi_p(\tau)/\varphi(\tau) \longrightarrow 1$ uniformly for $p \in J_1$, as $x \longrightarrow \infty$. Combining this observation with (3.11), we conclude that

$$E_2 = x + o(x). \tag{3.12}$$

To calculate $E_1$, we first consider the sums

$$S_{p_1,p_2} \stackrel{\text{def}}{=} \sum_{p_1|n} \sum_{p_2|n} e^{i\tau\left(T(n,p_1)-T(n,p_2)\right)}$$

for primes $p_1, p_2 \in \wp_x \cap J_1$. If $p_1 = p_2 = p$, then clearly we have $S_{p,p} = \frac{x}{p} + O(1)$. On the other hand, if $p_1 \neq p_2$, say $p_1 < p_2$, then, using Lemma 1 with $y = p_2$ and

$$f(q) = e^{i\tau\left(h(\frac{\log q}{\log p_1})-h(\frac{\log q}{\log p_2})\right)},$$

we get that

$$S_{p_1,p_2} = \frac{x}{p_1p_2} e^{i\tau h(\frac{\log p_1}{\log p_2})} \lambda_{p_1,p_2}(\tau) + O\left(\frac{x}{p_1p_2} \exp\left(-c_1\rho_1(x)\right)\right), \tag{3.13}$$

where

$$\lambda_{p_1,p_2}(\tau) = \prod_{q<p_2, q\neq p_1} \left(1 + \frac{e^{i\tau\left(h(\frac{\log q}{\log p_1})-h(\frac{\log q}{\log p_2})\right)} - 1}{q}\right).$$

A formula similar to (3.13) can easily be obtained in the case $p_1 > p_2$. Now define $S(x)$ so that $\log S(x) = \sqrt{\xi(\wp_x)}$. We now write

$$W \stackrel{\text{def}}{=} \{(p_1,p_2) \in \wp_x \times \wp_x\} = W_1 \cup W_2,$$

where

$$W_1 = \{(p_1,p_2) : p_1 < p_2 < p_1^{S(x)} \text{ or } p_2 < p_1 < p_2^{S(x)}\}$$

and

$$W_2 = W \setminus W_1.$$

If $(p_1,p_2) \in W_2$, $p_1 < p_2$, say, then, using Lemma 4, with $y_1(x) = p_1$, $y_2(x) = p_2$ and $r(x) = \min\left(\log u(x), \frac{\log x}{\log v(x)}, S(x)\right)$, we get that

$$\left|\frac{\lambda_{p_1,p_2}(\tau)}{|\varphi(\tau)|^2} - 1\right| \leq \rho\left(r(x)\right).$$

Hence we get that

$$\begin{aligned} E_1 = {} & \frac{1}{\xi(\wp_x)^2|\varphi(\tau)|^2}\left(\sum_p S_p + \sum_{(p_1,p_2)\in W_1, p_1\neq p_2} S_{p_1,p_2}\right) \\ & + \frac{x}{\xi(\wp_x)^2} \sum_{(p_1,p_2)\in W_2} \frac{1}{p_1p_2} + O\left(x\rho\left(r(x)\right)\right) \\ & + O\left(\frac{x}{\xi(\wp_x)^2} \sum \frac{1}{p_1} \sum_{p_1<p_2<p_1^{S(x)}} \frac{1}{p_2}\right) + O(xe^{-c_1\rho_1(x)}). \end{aligned}$$

Since $\sum_{p_1<p_2<p_1^{S(x)}} \frac{1}{p_2} \ll \log S(x)$, it follows that

$$\lim_{x\to\infty} \frac{1}{\xi(\wp_x)^2} \sum_{(p_1,p_2)\in W_1} \frac{1}{p_1p_2} = 0.$$

On the other hand, it is clear that $S_{p_1,p_2} \ll \frac{x}{p_1p_2}$ if $p_1 \neq p_2$ and furthermore that

$$\lim_{x\to\infty} \frac{1}{\xi(\wp_x)^2} \sum_{(p_1,p_2)\in W_2} \frac{1}{p_1p_2} = 1.$$

Hence it follows that

$$E_1 = x + o(x). \tag{3.14}$$

Substituting (3.12) and (3.14) in (3.10), we obtain (3.9). This completes the proof of Theorem 3.

4. **On the highest accumulation of prime divisors.** Let $\mathcal{X}$, $\wp_x$ $(x \in \mathcal{X})$ be as in Section 1 and let $\Delta > 1$. We shall assume that $\xi(\wp_x) \longrightarrow \infty$ as $x \longrightarrow \infty$. For each $y$ such that $y^{1/\Delta} \geq 2$, let $\alpha(n, y)$ be the number of distinct prime divisors $q$ of $n$ which are located in the open interval $(y^{1/\Delta}, y)$. Further, for each $n \leq x$, set

$$k(n) \stackrel{\text{def}}{=} \max_{p|n, p\in\wp_x} \alpha(n, p). \tag{4.1}$$

Our goal is to provide a precise estimate for $k(n)$.

Let $z_x^* = z$ be the solution of the equation

$$\frac{\Delta\xi(\wp_x)(\log\Delta)^z}{\Gamma(z+1)} = 1, \tag{4.2}$$

where $\Gamma$ is the Gamma function. Finally set $K_x = [z_x^*]$.

THEOREM 4. *Let $x_m$ be a subsequence of $\mathcal{X}$ for which, as $z \longrightarrow \infty$, both*

$$\frac{K_{x_m}!}{\Gamma(z^*+1)} \longrightarrow 0 \text{ and } \frac{\Gamma(z^*+1)}{(K_{x_m}+1)!} \longrightarrow 0 \tag{$*$}$$

*hold simultaneously (with $K_{x_m} = [z_{x_m}^*]$). Then*

$$\lim_{m\to\infty} \nu_{x_m}\{n \leq x_m : k(n) = K_{x_m}\} = 1. \tag{4.3}$$

*Without the assumption $(*)$, we have that, if $T_x$ is the closest integer to $z_x^*$, then*

$$\lim_{x\to\infty} \nu_x\{n \leq x : T_x - 1 \leq k(n) \leq T_x\} = 1. \tag{4.4}$$

REMARK. Taking into account (4.2), it follows from Theorem 4 that, for all but $o(x)$

integers $n \le x$, we have

$$k = k(n) \sim \frac{\log \xi(\wp_x)}{\log\log \xi(\wp_x)}.$$

PROOF. We divide the proof into two parts.

PART I. Given an integer $\ell \ge 1$ and a real number $y \ge 2$, let $Q_{y,\ell}$ be an arbitrary integer which is a product of $\ell$ distinct primes, $Q_{y,\ell} = q_1 q_2 \dots q_\ell$, such that $y^{1/\Delta} \le q_1 < q_2 < \cdots < q_\ell < y$. It is known that

$$\prod_{y^{1/\Delta}<p<y} \left(1 - \frac{1}{p}\right) = \frac{1}{\Delta} + O\left(e^{-\sqrt{\frac{\log y}{\Delta}}}\right) \tag{4.5}$$

and

$$\sum_{y^{1/\Delta}<p<y} \frac{1}{p} = (\log \Delta)\left(1 + O\left(e^{-\sqrt{\frac{\log y}{\Delta}}}\right)\right). \tag{4.6}$$

Actually for our purposes, more crude estimates will be enough.

Let $\ell = T_x + 1$. If for some integer $n \le x$, we have $k(n) \ge \ell$, then it must have a divisor $pQ_{p,\ell}$, where $p \in \wp_x$. Therefore

$$\nu_x\{n \le x : k(n) \ge \ell\} \le \sum_{p\in\wp_x} \frac{1}{p} \sum_{Q_{p,\ell}} \frac{1}{Q_{p,\ell}}. \tag{4.7}$$

Clearly we have

$$\sum_{Q_{p,\ell}} \frac{1}{Q_{p,\ell}} < \frac{1}{\ell!}\left(\sum_{p^{1/\Delta}<q<p} \frac{1}{q}\right)^{\ell},$$

the right hand side of which is, by (4.6),

$$\ll \frac{1}{\ell!}(\log \Delta)^{\ell}\left(1 + O\left(e^{-\sqrt{\frac{\log p}{\Delta}}}\right)\right)^{\ell}.$$

Since $\ell \sim \frac{\log \xi(\wp_x)}{\log\log \xi(\wp_x)}$, it follows that $\left(1 + O\left(e^{-\sqrt{\frac{\log p}{\Delta}}}\right)\right)^{\ell} \ll 1$ if $\log p \ge \Delta\left(\log\log \xi(\wp_x)\right)^2$. The contribution of the small primes $p$, that is those which satisfy $\log p < \Delta\left(\log\log \xi(\wp_x)\right)^2$ to the right hand side of (4.7) is

$$\ll \frac{1}{\ell!}(\log \Delta)^{\ell} e^{c\ell} \sum \frac{1}{p} \ll o(1)$$

as $x \longrightarrow \infty$. Here $c$ is a suitable positive constant satisfying $1 + O\left(e^{-\sqrt{\frac{\log p}{\Delta}}}\right) \le e^c$. Thus the right hand side of (4.7) becomes

$$\ll \frac{\xi(\wp_x)}{\ell!}(\log \Delta)^{\ell} + o_x(1).$$

This implies that

$$\nu_x\{n \le x : k(n) \ge T_x + 1\} = o_x(1) \quad (x \longrightarrow \infty).$$

Assume now that conditions $(*)$ holds. Then, by setting $\ell = K_{x_m} + 1$ and repeating the same argument as the one above, we conclude that

$$\lim_{m \longrightarrow \infty} \nu_{x_m}\{n \le x_m : k(n) > K_{x_m}\} = 0.$$

To prove that $k(n) \ge K_{x_m}$ and $k(n) \ge T_x - 1$ hold for almost all $n$ in (4.3) and (4.4), we shall ignore some elements of $\wp_x$, generate an appropriate subset $\wp_x'' \subset \wp_x$ and prove that

$$k''(n) \stackrel{\text{def}}{=} \max_{\substack{p|n \\ p \in \wp_x''}} \alpha(n,p) \tag{4.9}$$

satisfies $k''(n) \ge K_{x_m}$ and $k''(n) \ge T_x - 1$ for almost all $n$.

We set $C = C_1 \cup C_2$, where $C_1$ is made up of the first $t$ smallest elements $q_j \in \wp_x$ which satisfy

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_t} \in \left[\sqrt{\xi(\wp_x)}, \sqrt{\xi(\wp_x)} + 1\right],$$

and where $C_2$ is made up of the $s$ largest elements $q_j \in \wp_x$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_s} = \sqrt{\xi(\wp_x)} + O(1).$$

With this definition of $C$, define $\wp_x{}' = \wp_x \setminus C$. We shall now remove from $\wp_x{}'$ some "unwanted" elements, namely those $p_2 \in \wp_x{}'$ such that there exists a $p_1 \in \wp_x{}'$ such that

$$\left|\log \frac{\log p_2}{\log p_1}\right| < \frac{1}{\log p_2} \text{ or } \left|\log \frac{\Delta \log p_1}{\log p_2}\right| < \frac{1}{\log p_2};$$

clearly $\sum_{\{p_2\}} \frac{1}{p_2} = o(1)$ as $x \longrightarrow \infty$. We denote by $\wp_x{}''$ the set of uncancelled elements of $\wp_x{}'$. Hence we have $\xi(\wp_x{}') = \xi(\wp_x{}'') + o(1)$. Now if $p \in \wp_x{}''$, then

$$e^{\frac{1}{2}\sqrt{\xi(\wp_x)}} < \log p \text{ and } p < x^{e^{-\frac{1}{2}\sqrt{\xi(\wp_x)}}}.$$

Let $\Pi_p \stackrel{\text{def}}{=} \sum_{p^{1/\Delta} < q < p} \frac{1}{q}$. It is easy to see that

$$\sum \frac{1}{Q_{p,\ell}} = \frac{1}{\ell!}\Pi_p^\ell - \sigma_{p,\ell} \tag{4.10}$$

with

$$0 \le \sigma_{p,\ell} < \frac{\ell^2}{p^{1/\Delta}} \cdot \frac{1}{\ell!} \cdot \Pi_p^{\ell-1}$$

(see Halberstam and Roth [13]). We now choose $\ell$ in such a way that, as $x \longrightarrow \infty$,

$$\frac{\xi(\wp_x)}{\ell!}(\log\Delta)^{\ell} \longrightarrow \infty \text{ and } \frac{\xi(\wp_x)}{(\ell+1)!}(\log\Delta)^{\ell+1} = O(1).$$

Then clearly we also have that, as $x \longrightarrow \infty$,

$$\frac{\xi(\wp_x'')(\log\Delta)^{\ell}}{\ell!} \longrightarrow \infty,$$

and furthermore that

$$\ell \sim \frac{\log\xi(\wp_x'')}{\log\log\xi(\wp_x'')}.$$

PART II. First we let $U(n) = \#\{p : p \in \wp_x'', p \mid n, \alpha(n,p) = \ell\}$ and set

$$E = E(x) \stackrel{\text{def}}{=} \frac{\xi(\wp_x'')(\log\Delta)^{\ell}}{\ell!\,\Delta} \text{ and } D = D(x) \stackrel{\text{def}}{=} \sum_{n\le x}\big(U(n) - E\big)^2.$$

We proceed to estimate $D$ by using Turan's squaring method. Write

$$D = S_1 - 2ES_0 + E^2[x], \text{ where } S_0 = \sum_{n\le x} U(n) \text{ and } S_1 = \sum_{n\le x} U^2(n).$$

Clearly

$$\sum_{n\le x} U(n) = \sum_{p\in\wp_x''} \textstyle\sum_p,$$

where $\sum_p$ stands for the number of positive integers $n \le x$ that can be written as $n = Q_{p,\ell}pr$, where $q \not| r$ if $p^{1/\Delta} < q < p$ and $q \not| Q_{p,\ell}$. Since

$$\prod_{q\mid Q_{p,\ell}}\left(1 - \frac{1}{q}\right) = 1 + O\Big(\sum_{q\mid Q_{p,\ell}}\frac{1}{q}\Big) = 1 + O\left(\frac{\ell}{p^{1/\Delta}}\right) = 1 + o_x(1),$$

it follows, by using the sieve formula of Lemma 1, that

$$\textstyle\sum_p = \displaystyle\sum_{Q_{p,\ell}} \frac{x}{pQ_{p,\ell}} \prod_{p^{1/\Delta}<q<p}\left(1 - \frac{1}{q}\right)\left(1 + O\Big(e^{-c_1\frac{\log x/p}{\log(pQ_{p,\ell})}}\Big)\right).$$

Hence using (4.10), (4.5) and (4.6), we get that

$$S_0 = E\big(1 + o(1)\big)x. \tag{4.11}$$

Now

$$S_1 = \sum_{p_1,p_2\in\wp_x''} \textstyle\sum_{p_1,p_2},$$

where

$$\textstyle\sum_{p_1,p_2} = \displaystyle\sum_{\alpha(p_1,n)=\ell,\alpha(p_2,n)=\ell} 1.$$

Further define

$$\sum\nolimits_1 = \sum\nolimits_1^{(0)} + 2\sum\nolimits_1^{(1)} + 2\sum\nolimits_1^{(2)},$$

where

$$\sum\nolimits_1^{(0)} = \sum_p \sum\nolimits_{p,p}; \quad \sum\nolimits_1^{(1)} = \sum_{p_2} \sum_{p_2^{1/\Delta} < p_1 < p_2} \sum\nolimits_{p_1,p_2}; \quad \sum\nolimits_1^{(2)} = \sum_{p_2} \sum_{p_1 < p_2^{1/\Delta}} \sum\nolimits_{p_1,p_2}.$$

It is clear that

$$\sum\nolimits_1^{(0)} = S_0 = O(Ex).$$

We now proceed to estimate $\sum_1^{(1)}$. If $\alpha(n, p_1) = \ell$, $\alpha(n, p_2) = \ell$, then $p_1 p_2 | n$ and in both of the intervals $(p_1^{1/\Delta}, p_1)$, $(p_2^{1/\Delta}, p_2)$, $n$ contains exactly $\ell$ distinct prime divisors. Clearly $p_2[Q_{p_2,\ell}, Q_{p_1,\ell}]|n$ (here $[a, b]$ denotes the least common multiple of $a$ and $b$). Furthermore $[Q_{p_2,\ell}, Q_{p_1,\ell}] = Q_{p_2,\ell}R$, where $R|n$, and all the prime factors of $R$ are located in $(p_1^{1/\Delta}, p_2^{1/\Delta})$, and $R = 1$ or $\omega(R) \leq \ell - 1$. Observe that the conditions $\alpha(n, p_2) = \ell$, $R|n$ are clearly independent. Thus

$$\sum\nolimits_1^{(1)} \ll \sum_{p_2 Q_{p_2,\ell}} \frac{x}{p_2 Q_{p_2,\ell}} \prod_{p_2^{1/\Delta} < q < p_2} \left(1 - \frac{1}{q}\right) \sum_R \frac{1}{R}. \tag{4.12}$$

But, since $p_2^{1/\Delta} < p_1$, the interval $(p_2^{1/\Delta^2}, p_2^{1/\Delta})$ is certainly wider than the interval $(p_1^{1/\Delta}, p_2^{1/\Delta})$; hence

$$\sum_R \frac{1}{R} \leq 1 + \sum_{j=1}^{\ell-1} \frac{1}{j!} \Bigg( \sum_{p_2^{1/\Delta^2} < q < p_2^{1/\Delta}} \frac{1}{q} \Bigg)^j \ll 1, \tag{4.13}$$

Substituting (4.13) in (4.12), we conclude that

$$\sum\nolimits_1^{(1)} \leq cEx.$$

It remains to estimate $\sum_1^{(2)}$. First observe that, in this case, the intervals $[p_1^{1/\Delta}, p_1)$ and $[p_2^{1/\Delta}, p_2)$ are disjoint. Therefore

$$\sum\nolimits_{p_1,p_2} = \big(1 + o(1)\big) \sum \frac{x}{p_1 Q_{p_1,\ell} p_2 Q_{p_2,\ell}} \prod_{p_1^{1/\Delta} < q < p_1} \left(1 - \frac{1}{q}\right) \prod_{p_2^{1/\Delta} < q < p_2} \left(1 - \frac{1}{q}\right).$$

Summing up for $p_1$ and $p_2$, we have that

$$\sum\nolimits_1^{(2)} = \big(1 + o(1)\big)Ax + o(x),$$

where

$$A = \frac{1}{\Delta^2} \sum_{p_1 < p_2} \frac{1}{p_1 p_2} \sum \frac{1}{Q_{p_1,\ell}} \frac{1}{Q_{p_2,\ell}}.$$

Clearly we have that

$$2A \leq \frac{1}{\Delta^2}\Big(\sum_p \frac{1}{p}\Big(\sum \frac{1}{Q_{p,\ell}}\Big)\Big)^2.$$

But, we have shown earlier that the right hand side is $\big(1+o(1)\big)E^2$ as $x \longrightarrow \infty$. Hence we have, as $x \longrightarrow \infty$,

$$\sum\nolimits_1 \leq \big(1+o(1)\big)E^2 x.$$

We conclude from this that

$$0 \leq D \leq o(1)E^2 x,$$

and therefore that

$$\frac{1}{x}\#\big\{n \leq x : U(n) \neq \big(1+o(1)\big)E\big\} = o(x).$$

This completes the proof of Theorem 4.

5. **On** $\max_{p|n,p\in\wp_x} T(n,p)$**.** Using essentially the same reasoning as the one displayed in Section 4, we now prove two theorems.

THEOREM 5. *Let* $0 < a < 1$ *and let* $h\colon [0,1] \longrightarrow \mathbf{R}$ *be such that* $h(u) = 0$ *in* $[0,a)$ *and that* $\max_{a\leq u\leq 1} h(u) = M$ *exists and that* $M > 0$*; assume also that h attains its maximum at* $u = \lambda$ *and that it is continuous at* $\lambda$*. If* $\wp_x$ *is a set of primes* $p \leq x$*, then*

$$k^*(n) \stackrel{\text{def}}{=} \max_{p|n,p\in\wp_x} \sum_{q|n,\,q<p} h\Big(\frac{\log q}{\log p}\Big) = M\big(1+o(1)\big)\frac{\log\xi(\wp_x)}{\log\log\xi(\wp_x)}$$

*for all but* $o(x)$ *integers* $n \leq x$*, assuming that* $\xi(\wp_x) \longrightarrow \infty$*.*

PROOF. Choose $\varepsilon > 0$ and then $\delta > 0$ such that $h(u) \geq M - \varepsilon$ in $[\lambda-\delta,\lambda]$. For every $x$, let $k = k(x) = [z(x) - 1]$, where $z(x)$ is the positive solution of

$$\xi(\wp_x)\Big(\frac{\delta}{\lambda}\Big)^z = \Gamma(z+1).$$

For each prime $p|n$, let $\gamma(n,p) = 1$ if $p \in \wp_x$ and if there are exactly $k$ prime divisors of $n$ located in $[p^{\lambda-\delta}, p^{\lambda})$ and no other prime divisor in $(p^a, p)$; otherwise set $\gamma(n,p) = 0$. One can see, using the same techniques as in Section 4, that, for almost all $n$, $\sum_{p|n,p\in\wp_x}\gamma(n,p) \geq 1$. But then

$$k^*(n) \geq (M-\varepsilon)k. \tag{5.1}$$

Using the remark following Theorem 4, we have that

$$k \sim \frac{\log\xi(\wp_x)}{\log\log\xi(\wp_x)}.$$

Set

$$K \stackrel{\text{def}}{=} \Big[(1+\varepsilon')\frac{\log\xi(\wp_x)}{\log\log\xi(\wp_x)}\Big]$$

where $\varepsilon' > 0$ is an arbitrary constant. We shall prove that the number of integers $n \le x$ for which $n$ has at least $K$ prime divisors in a suitable interval $[p^a, p]$ where $p|n$ and $p \in \wp_x$ is $o(x)$.

For this, we first let $y$ be defined by

$$\log\log y = \left(1 + \frac{\varepsilon'}{2}\right) \frac{\log \xi(\wp_x)}{\log\log \xi(\wp_x)}.$$

By the Turan-Kubilius inequality, there exist at most $o(x)$ integers $n \le x$, which have at least $K$ prime divisors up to $y$. The other integers $n$ have at least one divisor $pQ_{p,K}$ where $p > y$, $p \in \wp_x$ and all prime factors of $Q_{p,K}$ are located in $[p^a, p)$. Their number is

$$\ll \sum_{n \le x} \sum_{\substack{pQ_{p,K}|n \\ p\in\wp_x, p>y}} 1 \le \frac{x}{K!} \sum_{p\in\wp_x, p>y} \frac{1}{p}\left(\sum_{p^a<q<p} \frac{1}{q}\right)^K$$

$$\ll \frac{x}{K!}\left(\log \frac{1}{a}\right)^K \sum_{p\in\wp_x, p>y} \frac{1}{p}\left(1 + e^{-\sqrt{\log p}}\right)^K \ll \frac{x\xi(\wp_x)(\log 1/a)^K}{K!}.$$

But this last expression is $o(x)$ as $x \longrightarrow \infty$. Hence it is clear that $k^*(n) \le MK$ for all but $o(x)$ integers $n \le x$. Combining this with (5.1), the theorem follows.

THEOREM 6. *Let $\wp_x$ be a "large set" of primes in the sense that*

$$\lim_{x\to\infty} \frac{\log \xi(\wp_x)}{\log\log\log x} = 1.$$

*Let $h$: $[0, 1] \longrightarrow \mathbf{R}$ be such that $|h(u)|$ is monotonic, and assume that* $\max_{0\le u\le 1} h(u) = M > 0$ *exists, that it is attained at $u = \lambda$ and that $h$ is continuous at $\lambda$. Let $k^*(n)$ be defined as in Theorem 5. Then, for all but $o(x)$ integers $n \le x$,*

$$k^*(n) = M\big(1 + o(1)\big)\frac{\log_3 n}{\log_4 n}. \tag{5.2}$$

*(Here* $\log_\ell n$ *stands for the $\ell$-th iterative of* $\log n$.*)*

PROOF. From the integrability and monotonicity of $|h|$ it follows that $\frac{|h(\delta u)|}{|h(u)|} \longrightarrow 0$ as $\delta \longrightarrow 0$ uniformly in some interval $[0, \varepsilon_1]$. Let

$$t(\delta) \overset{\text{def}}{=} \max_{0\le u\le\varepsilon_1} \left|\frac{h(\delta u)}{h(u)}\right|.$$

Let $\varepsilon_2$ be a small positive number to be specified later and let

$$h_1(u) \overset{\text{def}}{=} \begin{cases} |h(u)| & \text{if } u \in [0, \varepsilon_2], \\ 0 & \text{if } u > \varepsilon_2. \end{cases}$$

Let

$$K^*(n) = \max_{p|n} \sum_{q|n, q<p} h_1\left(\frac{\log q}{\log p}\right).$$

where the maximum is now taken on all prime divisors $p$ of $n$. Define $T_x = (1+\varepsilon_x)\frac{\log_3 x}{\log_4 x}$, where $\varepsilon_x \longrightarrow 0$ as $x \longrightarrow \infty$. With a proper choice of $\varepsilon_x$ and using Theorem 4, we can state that, for almost all integers $n \le x$, $n$ contains no more than $T_x$ prime factors in an interval $[y^\delta, y]$ for some $y$. Therefore

$$
\begin{aligned}
K^*(n) &\le T_x\big(h_1(\varepsilon_2) + h_1(\delta\varepsilon_2) + h_1(\delta^2\varepsilon_2) + \cdots\big) \\
&\le T_x h_1(\varepsilon_2)(1 + t(\delta) + t^2(\delta) + \cdots) \\
&\le 2T_x h_1(\varepsilon_2).
\end{aligned} \tag{5.3}
$$

Now let

$$h_2(n) \stackrel{\text{def}}{=} \begin{cases} h(u) & \text{if } u \in [\varepsilon_2, 1], \\ 0 & \text{if } u < \varepsilon_2. \end{cases}$$

If we further set

$$k_1(n) = \max_{p|n, p\in\wp_x} \sum_{q|n, q<p} h_2\left(\frac{\log q}{\log p}\right),$$

we note that we have already proved (Theorem 5) that

$$k_1(n) = M\big(1+o(1)\big)\frac{\log_3 x}{\log_4 x}.$$

But it is obvious that

$$k_1(n) - K^*(n) \le k^*(n) \le k_1(n) + K^*(n).$$

Because of (5.3), if $\varepsilon_2$ is small enough, we have that $K^*(n) = o\big(\frac{\log_3 x}{\log_4 x}\big)$. This allows us to conclude that (5.2) is true and hence this finishes the proof of Theorem 6.

**6. The distribution of $T(n, X)$ in the case $h(v) = v^\beta$.** Let $h(v) = v^\beta$, $\beta > 0$. Let $\tau > 0$ and recall that in this case we have

$$\alpha(\tau) = \frac{1}{\beta}\int_0^\tau \frac{e^{iv}-1}{v}\,dv, \quad \varphi(\tau) = \exp\big(\alpha(\tau)\big).$$

Since $\Re\big(\alpha(\tau)\big) = O(1) + \frac{1}{\beta}\int_1^\tau \frac{\cos v - 1}{v}\,dv$ and $\int_1^\tau \frac{\cos v}{v}\,dv$ is bounded, it follows that, as $\tau \longrightarrow \infty$,

$$\Re\big(\alpha(\tau)\big) = -\frac{1}{\beta}\log\tau + O(1),$$

and therefore

$$|\varphi(\tau)| \le c_1|\tau|^{-1/\beta} \tag{6.1}$$

holds.

Let $F(z)$ be the distribution function which corresponds to $|\varphi(\tau)|$. By using Lemma 5 and (6.1), we easily get that

(a) in the case $\beta < 1$, $F(z)$ is absolutely continuous and has a bounded derivative,
(b) in the case $\beta > 1$, $Q_F(h) \ll h^{1/\beta}$ and $S_F(h) \ll h^{1/\beta}$.

The case $\beta = 1$ has already been considered by Bovey[1].

Let $\varphi_x(\tau)$ be as in (2.7) and set $h(u) = u^\beta$. We shall now estimate

$$\left|\frac{\varphi_x(\tau)}{\varphi(\tau)} - 1\right| \tag{6.2}$$

in the interval $|\tau|\left(\frac{\log 2}{\log x}\right)^\beta < \pi - \Delta$, where $\Delta > 0$ is fixed.

In order to simplify the notation, let $h_q = \left(\frac{\log q}{\log x}\right)^\beta$. Further set

$$z \overset{\text{def}}{=} \begin{cases} x & \text{if } |\tau| \le \frac{1}{2}, \\ \exp\left(\left(\frac{1}{2|\tau|}\right)^{1/\beta}\log x\right) & \text{if } |\tau| > \frac{1}{2} \end{cases}$$

and write

$$\varphi_x(\tau) = \varphi_x^{(1)}(\tau)\varphi_x^{(2)}(\tau),$$

where

$$\varphi_x^{(1)}(\tau) = \prod_{q\le z}\left(1 + \frac{e^{i\tau h_q} - 1}{q}\right), \quad \varphi_x^{(2)}(\tau) = \prod_{z<q\le x}\left(1 + \frac{e^{i\tau h_q} - 1}{q}\right).$$

Let

$$\alpha_1(\tau) = \int_0^{\frac{\log z}{\log x}} \frac{e^{i\tau v^\beta} - 1}{v}\,dv, \quad \alpha_2(\tau) = \int_{\frac{\log z}{\log x}}^1 \frac{e^{i\tau v^\beta} - 1}{v}\,dv.$$

We have

$$\log\varphi_x^{(1)}(\tau) = \sum_{q\le z}\log\left(1 + \frac{e^{i\tau h_q} - 1}{q}\right) = \sum_{q\le z}\frac{e^{i\tau h_q} - 1}{q} + O(A_z), \tag{6.3}$$

where

$$A_z = \sum_{q\le z}\frac{|e^{i\tau h_q} - 1|}{q^2}. \tag{6.4}$$

We have, by using the prime number theorem in the form $R(u) = \pi(u) - \mathrm{Li}(u) \ll u\exp(-(\log u)^{1/2})$, that

$$\begin{aligned}\sum_{q\le z}\frac{e^{i\tau h_q} - 1}{q} &= \int_2^z \frac{e^{i\tau h_u} - 1}{u}\,d\mathrm{Li}(u) + \int_2^z \frac{e^{i\tau h_u} - 1}{u}\,dR(u) \\ &= \alpha_1(\tau) + J,\end{aligned}$$

say, where $J = J(z)$.

We now estimate the integral $J$. Set $J_1 = \Re J$ and $J_2 = \Im J$. Then $|J| \le |J_1| + |J_2|$, and

$$J_\nu = \int_2^z \frac{g_\nu(u)}{u}\,dR(u),$$

where $g_1(u) = 1 - \cos\left(\tau\left(\frac{\log u}{\log x}\right)^\beta\right)$, $g_2(u) = 1 - \sin\left(\tau\left(\frac{\log u}{\log x}\right)^\beta\right)$.

Observing that $g'_\nu(u)$ ($\nu = 1, 2$) have constant signs on $[2, z]$, one can prove that

$$|J| \leq \frac{c_1|\tau|}{(\log x)^\beta}. \tag{6.5}$$

Indeed, integrating by parts, we obtain

$$\begin{aligned} J_\nu &= \frac{g_\nu(u)}{u}R(u)\Big|_2^z - \int_2^z R(u)\left(\frac{g'_\nu(u)}{u} - \frac{g_\nu(u)}{u^2}\right) du \\ &\ll \left|\frac{g_\nu(z)}{z}R(z)\right| + |g_\nu(2)| + \int_2^z \frac{|g_\nu(u)|}{u} e^{-(\log u)^{1/2}} du \\ &\quad + \left|\int_2^z e^{-(\log u)^{1/2}} g'_\nu(u)\, du\right|. \end{aligned}$$

Using one more time partial integration, one can see that this last integral is less than

$$|g_\nu(z)|e^{-(\log z)^{1/2}} + |g_\nu(2)| + \left|\int_2^z g_\nu(u)\left(e^{-(\log u)^{1/2}}\right)' du\right|.$$

Furthermore, we have

$$|g_\nu(u)| \ll |\tau|\frac{(\log u)^\beta}{(\log x)^\beta},$$

and hence we obtain immediately that

$$J_\nu \ll \frac{|\tau|}{(\log x)^\beta},$$

which proves (6.5).

On the other hand, it is clear that

$$A_z \ll \frac{|\tau|}{(\log x)^\beta}.$$

Assume now that $|\tau| > \frac{1}{2}$. Define the sequence

$$z = u_0 < u_1 < u_2 < \cdots$$

by

$$\frac{\log u_k}{\log x} = \left(\frac{k\pi}{2|\tau|}\right)^{1/\beta} \qquad (k = 1, 2, \ldots).$$

Arguing as earlier, we have

$$\log \varphi_x^{(2)}(\tau) - \alpha_2(\tau) = \int_z^x \frac{e^{i\tau h_u} - 1}{u}\, dR(u) + O\Bigg(\sum_{z<q\leq x} \frac{|e^{i\tau h_u} - 1|}{q^2}\Bigg). \tag{6.6}$$

The error term is $\ll 1/z \log z$. Set $K = \max\{k : u_k < x\}$ and modify $u_{K+1}$ to be $x$. Then write

$$\int_z^x \frac{e^{i\tau h_u} - 1}{u}\, dR(u) = \int_{u_0}^{u_1} + \cdots + \int_{u_{K-1}}^{u_K} + \int_{u_K}^x = I_0 + \cdots + I_K + I_{K+1}. \tag{6.7}$$

Further observe that the derivatives of the functions $g_\nu(u)$ ($\nu = 1, 2$) defined earlier have constant signs in each of the intervals $[u_0, u_1]$, $[u_1, u_2], \ldots, [u_{K-1}, u_K]$, $[u_K, x]$. For $j = 0, 1, \ldots, K$, write

$$I_j = I_j^{(1)} + iI_j^{(2)}, \text{ where } I_j^{(1)} = \Re I_j,\ I_j^{(2)} = \Im I_j.$$

Then, using integration by parts, we have, for each $j < K$, $\nu = 1, 2$,

$$I_j^{(\nu)} \ll e^{-(\log u_j)^{1/2}} + \left|\int_{u_j}^{u_{j+1}} R(u)\frac{g_\nu'(u)}{u}\,du\right| + \left|\int_{u_j}^{u_{j+1}} \frac{R(u)}{u^2} g_\nu(u)\,du\right|. \tag{6.8}$$

Since $g_\nu'(u)$ does not change its sign in $[u_j, u_{j+1}]$, we find, using integration by parts, that the second term on the right hand side of (6.8) is less than

$$e^{-(\log u_j)^{1/2}} + \int_{u_j}^{u_{j+1}} \left(e^{-(\log u)^{1/2}}\right)' g_\nu(u)\,du.$$

Since $|g_\nu(u)| \leq 1$, summing up for $j$, we easily obtain that

$$\sum_{j=0}^{K+1} I_j \ll \sum_{\nu=1,2}\left(\sum_j I_j^{(\nu)}\right) \ll \sum_j e^{-(\log u_j)^{1/2}} + \int_z^x \left(e^{-(\log u)^{1/2}}\right)' du$$
$$+ \int_z^x \frac{|R(u)|}{u^2}\left(|g_1(u)| + |g_2(u)|\right) du.$$

The first integral is less than $\exp\left(-(\log z)^{1/2}\right)$. Since $\log u_j > j^{1/\beta}\log u_1 > j^{1/\beta}\log u_0$, it follows that

$$\sum_j e^{-(\log u_j)^{1/2}} \ll e^{-(\log z)^{1/2}}.$$

To estimate the last integral, we observe that $|g_\nu(u)| \leq 1$, whence, since $|R(u)| \ll u\exp(-(\log u)^{1/2})$, we deduce that it is also $\ll e^{-(\log z)^{1/2}}$.

We have thus proven that

$$\log \varphi_x^{(2)}(\tau) - \alpha_2(\tau) \ll \frac{1}{z\log z}. \tag{6.9}$$

Clearly

$$\frac{1}{z\log z} \ll \frac{|\tau|}{(\log x)^\beta}.$$

Hence, collecting our inequalities, we get that

$$|\log \varphi_x(\tau) - \alpha(\tau)| \leq \frac{c_1|\tau|}{(\log x)^\beta} \tag{6.10}$$

uniformly for $|\tau|\left(\frac{\log 2}{\log x}\right)^\beta < \pi - \Delta$. Since

$$\left|\frac{\varphi_x(\tau)}{\varphi(\tau)} - 1\right| \leq \left|\exp\left(\log \varphi_x(\tau) - \alpha(\tau)\right) - 1\right| \ll \left|\log \varphi_x(\tau) - \alpha(\tau)\right|,$$

we get

$$|\varphi_x(\tau) - \varphi(\tau)| \le c_1 \frac{|\tau|}{(\log x)^\beta} |\varphi(\tau)| \tag{6.11}$$

uniformly for

$$|\tau| \left(\frac{\log 2}{\log x}\right)^\beta < \pi - \Delta. \tag{6.12}$$

REMARK. The inequality (6.11), in the case $\beta = 1$, has already been obtained by Bovey [1].

Let $0 < \theta \le 1$, where $\theta = \theta(X)$ satisfies $X^\theta \longrightarrow \infty$ as $X \longrightarrow \infty$. Let

$$H_{X,\theta}(z) \stackrel{\text{def}}{=} \frac{1}{X} \#\{n \le X, T(n, X^\theta) < z\} \tag{6.13}$$

and

$$\psi_{X,\theta}(\tau) \stackrel{\text{def}}{=} \frac{1}{X} \sum_{n \le X} e^{i\tau T(n, X^\theta)}. \tag{6.14}$$

We shall now approximate $H_{X,\theta}(z)$ by $F(z)$. To do this, we shall use Lemma 5, Lemma 1 and our inequalities (6.11) and (6.12).

First it is clear that

$$\begin{aligned} \psi_{X,\theta}(\tau) - 1 &= \frac{1}{X} \sum_{n \le X} \left(e^{i\tau T(n, X^\theta)} - 1\right) \\ &\ll |\tau| \sum_{q \le X^\theta} \left(\frac{\log q}{\log X^\theta}\right)^\beta \ll |\tau| \end{aligned}$$

and also that $|\varphi(\tau) - 1| \ll |\tau|$. Hence we obtain that

$$|\psi_{X,\theta}(\tau) - \varphi(\tau)| \ll |\tau|. \tag{6.15}$$

This inequality will be used in the range $0 \le |\tau| \le 1$. Applying Lemma 1 to the function $f(n) = e^{i\tau T(n, X^\theta)}$, we obtain that

$$|\psi_{X,\theta}(\tau) - \varphi_{X^\theta}(\tau)| \ll e^{-c_1/\theta}. \tag{6.16}$$

Hence, by (6.11) and (6.12), we get that

$$|\psi_{X,\theta}(\tau) - \varphi(\tau)| \ll e^{-c_1/\theta} + c_2 \frac{|\tau|}{(\log X)^\beta \theta^\beta} |\varphi(\tau)| \tag{6.17}$$

holds, if $|\tau| \le \theta^\beta \left(\frac{\log X}{\log 2}\right)^\beta \stackrel{\text{def}}{=} Q$, say. Now let $2 \le T \le Q$. From Lemma 5, we have

$$\begin{aligned} S &\stackrel{\text{def}}{=} \sup_z |H_{X,\theta}(z) - F(z)| \\ &\ll S_F(1/T) + \int_0^{e^{-c_1/\theta}} d\tau + \int_{e^{-c_1/\theta}}^T \left\{ e^{-c_1/\theta} + \frac{\tau}{Q} |\varphi(\tau)| \right\} \frac{d\tau}{\tau} \\ &\ll S_F(1/T) + (\theta^{-1} + \log T) e^{-c_1/\theta} + \frac{1}{Q} \int_1^T |\varphi(\tau)| d\tau, \end{aligned} \tag{6.18}$$

where $S_F(1/T)$ is defined in (2.8). Consequently, if $\beta > 1$, then

$$S \ll T^{-1/\beta} + (\theta^{-1} + \log T)e^{-c_1/\theta} + \frac{T^{1-1/\beta}}{Q}, \tag{6.19}$$

and for $\beta < 1$,

$$S \ll \frac{1}{T} + (\theta^{-1} + \log T)e^{-c_1/\theta} + \frac{1}{Q}, \tag{6.20}$$

because of the inequality $\varphi(\tau) \ll \tau^{-1/\beta}$. Clearly the last summand on the right hand side of both (6.19) and (6.20) can be cancelled, since the first summands are of larger order.

Suppose that $\beta > 1$. Assume that $X \geq 4$ and that $\left(\frac{\log X}{\log 2}\right)^\theta > e^{c_1}$. Set $T = \frac{e^{c_1\beta/\theta}}{\theta^\beta}$. Then the inequality $T \leq Q$ holds, and the right hand side of (6.19) is less than $\frac{1}{\theta}e^{-c_1/\theta}$.

This choice of $T$ is also allowed in the case $\beta < 1$ as well and thus leads to the inequality

$$S \ll \left(\frac{1}{\log X^\theta}\right)^\beta + \left[\log(\log X^\theta) + \frac{1}{\theta}\right]e^{-c_1/\theta}.$$

We have thus proven the following

THEOREM 7. *Let $h(u) = u^\beta$, $\beta \neq 1$, $X \geq 4$, $\theta = \theta(X)$ be such that $\theta \leq 1$ and that $\left(\frac{\log X}{\log 2}\right)^\theta > e^{c_1}$ holds (where $c_1 = c_1(\beta)$ is defined by (2.3)). Further let $H_{X,\theta}(z)$ be as in (6.13), $F(z)$ be the distribution function which corresponds to $\varphi(\tau)$. Then, with $S$ defined in (6.18), we have:*

- $S \leq c_2(\beta)\theta^{-1}e^{-c_1/\theta}$ *if* $\beta > 1$,
- $S \leq \frac{c_3(\beta)}{(\log X^\theta)^\beta} + c_4(\beta)\left[\log(\log X^\theta) + \frac{1}{\theta}\right]e^{-c_1/\theta}$ *if* $\beta < 1$.

**7. On the maximal gap between the prime factors.** In [8], Erdős proved that the density of the set of integers $n$ satisfying $\max_{1\leq i\leq\omega(n)-1}\frac{\log p_{i+1}(n)}{\log p_i(n)} > z\log\log n$ is $1 - \exp(-1/z)$.

Let $\mathcal{X}$ and $\wp_x$ $(x \in \mathcal{X})$ be as in Section 3, $h$ as in Lemma 3, and assume that

$$\lim_{x\to\infty} \xi(\wp_x) = +\infty. \tag{7.1}$$

We shall assume that $h$ is monotonically increasing in a neighbourhood of 0.

In this section, we are interested in the distribution of

$$\Upsilon(n) \stackrel{\text{def}}{=} \min_{p|n,\, p\in\wp_x,\, p>p(n)} T(n,p) = \min_{p|n,\, p\in\wp_x,\, p>p(n)} \sum_{q|n,\, q<p} h\left(\frac{\log q}{\log p}\right)$$

Let

$$H(v) \stackrel{\text{def}}{=} \int_0^v \frac{h(u)}{u}\,du$$

and assume that

$$H(v) \ll h(v). \tag{7.2}$$

From the existence of the integral $\int_0^1 \frac{h(u)}{u}\,du$ and from the monotonicity of $h$ in a neighbourhood of 0, we have that

$$\max_u \frac{h(\delta u)}{h(u)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \tag{7.3}$$

Additionally we shall assume that either

$$\lim_{u\to 0} \frac{H(u)}{h(u)} = 0 \tag{7.4}$$

or

$$H(u) \gg h(u) \tag{7.5}$$

holds.

Note that condition (7.4) implies that

$$\lim_{u\to 0} \frac{h(ru)}{h(u)} = 0 \text{ for every } 0 < r < 1. \tag{7.6}$$

Let $x \in \chi$ be given. Given an integer $n$ and $p$ a prime factor of $n$, let $q(n,p)$ be the largest prime factor of $n$ which is smaller than $p$. Further let

$$\ell_n \stackrel{\text{def}}{=} \min_{\substack{p\in\wp_x \\ p>p(n)}} \frac{\log q(n,p)}{\log p}. \tag{7.7}$$

LEMMA 6. *Let* $0 < z < \infty$. *Then*

$$\lim_{x\in\chi} \frac{1}{x}\#\{n \le x : \ell_n > z/\xi(\wp_x)\} = 1 - e^{-z}.$$

PROOF. The proof can be obtained in the same way as it was done by Erdős in [8].

Assume for the moment that (7.4) holds. Let $U_z$ be the set of those integers $n \le x$ for which

$$\Upsilon(n) \ge h\left(\frac{z}{\xi(\wp_x)}\right)$$

and $V_z$ be the set of those integers $n \le x$ for which $\ell_n > z/\xi(\wp_x)$. It is clear that $V_z \subset U_z$ and consequently that card $V_z \le$ card $U_z$. Furthermore, given a fixed $\varepsilon > 0$, we have that $U_z \subset V_{z-\varepsilon} \cup (\overline{V_{z-\varepsilon}} \cap U_z)$.

We first estimate card$(\overline{V_{z-\varepsilon}} \cap U_z)$. If $n \in \overline{V_{z-\varepsilon}} \cap U_z$, then

$$\Upsilon(n) \le \sum_{\substack{p\in\wp_x \\ p>p(n)}}{}^{*} \sum_{\substack{q|n \\ \frac{\log q}{\log p}<\rho_\varepsilon}} h\left(\frac{\log q}{\log p}\right), \quad \rho_\varepsilon = \frac{z-\varepsilon}{\xi(\wp_x)}$$

where $*$ indicates that we sum over those primes $p$ for which $\frac{\log q(n,p)}{\log p} < \rho_\varepsilon$ holds.

Now let us consider

$$S \stackrel{\text{def}}{=} \sum_{\substack{n \le x \\ n \in \overline{V_{z-\varepsilon}} \cap U_z}} \Upsilon(n).$$

Then, by the Eratosthenian sieve, we obtain that

$$\begin{aligned} S &\ll x \sum_{p \in \wp_x} \sum_{q < p^{\rho_\varepsilon}} \frac{1}{qp} h\left(\frac{\log q}{\log p}\right) \frac{\log q}{\log p} \\ &\ll x \sum_{p \in \wp_x} \frac{1}{p \log p} \int_1^{p^{\rho_\varepsilon}} h\left(\frac{\log y}{\log p}\right) \frac{\log y}{y} \, d\pi(y) \\ &\ll x \sum_{p \in \wp_x} \frac{1}{p \log p} \int_0^{\rho_\varepsilon \log p} h\left(\frac{t}{\log p}\right) dt = x\xi(\wp_x) \int_0^{\rho_\varepsilon} h(u)\, du \\ &< x\xi(\wp_x) h(\rho_\varepsilon)\rho_\varepsilon < xzh(\rho_\varepsilon). \end{aligned}$$

From (7.6) we have that

$$\frac{h(\rho_\varepsilon)}{h\big(z/\xi(\wp_x)\big)} \longrightarrow 0 \text{ as } x \longrightarrow \infty.$$

Consequently, $\Upsilon(n) > h\big(z/\xi(\wp_x)\big)$ implies that

$$\operatorname{card}(\overline{V_{z-\varepsilon}} \cap U_z) \le \frac{S}{h\big(z/\xi(\wp_x)\big)} = o(x) \text{ as } x \longrightarrow \infty.$$

Thus we have

$$\operatorname{card}(U_z) \le \operatorname{card}(V_{z-\varepsilon}) + \operatorname{card}(\overline{V_{z-\varepsilon}} \cap U_z) \le x(1 - e^{-z+\varepsilon}) + o(x).$$

Since $\varepsilon > 0$ is arbitrary, we obtain that

$$\frac{\operatorname{card}(U_z)}{x} = 1 - e^{-z} + o_x(1).$$

We have thus proved the following

THEOREM 8. *Let* $h\colon [0, 1] \longrightarrow \mathbf{R}$ *be increasing in a neighbourhood of zero. Assume that (7.4) holds. Let* $\wp_x$ *be a sequence of sets of primes such that* $\lim_{x\to\infty} \xi(\wp_x) = +\infty$. *Let* $0 < z < \infty$. *Then the number of integers* $n \le x$ *for which*

$$\Upsilon(n) > h\big(z/\xi(\wp_x)\big)$$

*holds is*

$$x\big(1 + o(1)\big)(1 - e^{-z}).$$

Hence from now on we shall assume that (7.5) holds.

One should expect the normalizing factor to be $h\big(1/\xi(\wp_x)\big)$, that is that

$$\frac{\Upsilon(n)}{h\big(1/\xi(\wp_x)\big)}$$

has a limit distribution.

Let $M_0(x)$ be the number of integers $n \le x$ such that

$$\Upsilon(n) \ge h\left(\frac{z}{\xi(\wp_x)}\right). \tag{7.8}$$

Here $z$ is an arbitrary but fixed positive number.

Let $N(x) = x - M_0(x)$ be the number of integers $n \le x$ for which (7.8) does not hold. Assume that $x$ is large. If for some integer $n \le x$ and some prime $p$ that divides $n$, $p \in \wp_x$, one has $T(n,p) < h\big(z/\xi(\wp_x)\big)$, then $n$ does not contain any prime divisors in the interval $[p^{z/\xi(\wp_x)}, p)$. But for a given prime $p$, the number of such integers $n \le x$ is clearly

$$\ll \frac{x}{p} \prod_{p^{z/\xi(\wp_x)}<q<p} \left(1 - \frac{1}{q}\right) \ll \frac{x}{p\xi(\wp_x)}.$$

Hence it follows that, when we count $N(x)$, we only make an error of order $o(x)$ if we ignore those integers $n$ for which $T(n,p) < h\big(z/\xi(\wp_x)\big)$ for some prime $p \in \wp_x^* \subset \wp_x$, where $\wp_x^*$ is such that $\lim_{x\to\infty} \frac{\xi(\wp_x^*)}{\xi(\wp_x)} = 0$.

We can easily construct such a set $\wp_x^*$. We let $\wp_x^*$ be the set made up of the smallest and the largest elements of $\wp_x$, that is, those primes $p \in \wp_x$ which also belong to $[1, y_x] \cup [w_x, x]$, where $y_x$, $w_x$ are determined by the equations

$$\log\log y_x = \frac{\xi(\wp_x)}{\log \xi(\wp_x)}, \quad \log \frac{\log x}{\log w_x} = \frac{\xi(\wp_x)}{\log \xi(\wp_x)}.$$

Let $\wp_x' = \wp_x \setminus \wp_x^*$ and denote by $N'(x)$ the number of integers $n \le x$ for which there exists $p \in \wp_x'$ such that $T(n,p) < h\big(z/\xi(\wp_x)\big)$. Let $p_1 < p_2 < \ldots < p_k$ be $k$ primes chosen from the set $\wp_x'$, and let

$$N(p_1, \ldots, p_k) \overset{\text{def}}{=} \big\{n \le x : p_1 \ldots p_k \mid n \text{ and } T(n, p_j) < h\big(z/\xi(\wp_x)\big), j = 1, \ldots, k\big\}.$$

Further set, for each $k \in \mathbf{N}$,

$$N_k(x) \overset{\text{def}}{=} \sum_{p_1<\ldots<p_k} N(p_1, \ldots, p_k).$$

Then, by the inclusion-exclusion process, we have that

$$N'(x) = N_1(x) - N_2(x) + N_3(x) - \cdots$$

and the sum of the first $k$ terms on the right hand side is $\ge N'(x)$ if $k$ is even, and $\le N'(x)$ if $k$ is odd.

We now estimate $N(p_1, \ldots, p_k)$. To simplify the notation, write $w = w_x = z/\xi(\wp_x)$. If, for each $j = 1, \ldots, k$, we have $p_j | n$ and $T(n, p_j) < h(w)$, then $n$ does not have any prime divisors in the intervals $(p_j^w, p_j)$. This clearly implies that, for $k \geq 2$, one has

$$p_j < p_{j+1}^w \quad (j = 1, \ldots, k-1)$$

Using this and (2.2), we have that

$$N(p_1, \ldots, p_k) \ll \sum_{m \leq \frac{x}{p_1 \cdots p_k}, p(m) > 2^{1/w^k}} 1 = \Phi\Big(\frac{x}{p_1 \cdots p_k}, 2^{1/w^k}\Big) \tag{7.9}$$

$$\ll \frac{x}{p_1 \cdots p_k} \frac{1}{\log 2^{1/w^k}} \ll \frac{x}{p_1 \cdots p_k} w^k$$

We shall allow $k$ to run from 1 to $K_x$, where $K_x \longrightarrow +\infty$ as slowly that $K_x \log w_x \longrightarrow 0$ as $x \longrightarrow \infty$ and we will choose another variable $R_x$ (which also tends to $+\infty$ as $x \longrightarrow \infty$) in such a way that

$$K_x^2 (\log R_x) w_x = o(1). \tag{7.10}$$

This will permit us to show that

$$S \stackrel{\text{def}}{=} \sum_{k=1}^{K_x} \sum{}' N(p_1, \ldots, p_k) = o(x), \tag{7.11}$$

where $\sum'$ runs over all collections $p_1 < \ldots < p_k$ ($p_j \in \wp_x'$, $j = 1, \ldots, k$) for which there exist at least two primes $p_i < p_{i+1}$ close to one another, in the sense that $p_i^{R_x} > p_{i+1}$. Since $\sum_{Q<q<Q^{R_x}} \frac{1}{q} \ll \log R_x$, it follows, using (7.9), that $\sum' \ll x(\log R_x) w$. Therefore

$$S = O\big(K_x^2 (\log R_x) w x\big) = o(x),$$

which proves (7.11). In order that (7.10) be satisfied, we choose

$$R_x = \exp(1/\sqrt{w}). \tag{7.12}$$

Because of (7.11), we may assume that the prime divisors of $n$ are far apart in the sense that $p_i < p_{i+1}^{1/R_x}$ for $i = 1, \ldots, k-1$.

For such collection of primes $p_1 < \cdots < p_k$ (that is, satisfying $p_i < p_{i+1}^{1/R_x}$), we consider the expressions

$$A_{p_1, \ldots, p_k}(\tau_1, \ldots, \tau_k) \stackrel{\text{def}}{=} \sum_{n \leq x}{}^{*} \exp\Big\{ i \Big( \sum_{j=1}^{k} \tau_j T(n, p_j) \Big) \Big\}$$

where the $*$ in the sum indicates that it runs over those integers $n \leq x$ which are divisible by $p_1, \ldots, p_k$ but which do not contain any prime divisors in the intervals $(p_j^w, p_j)$ $(j = 1, \ldots, k)$. Then, by the sieve formula, we get, as $x \longrightarrow \infty$,

$$A_{p_1, \ldots, p_k}(\tau_1, \ldots, \tau_k) = \frac{x w^k}{p_1 \cdots p_k} \exp\{ i C(\tau_1, \ldots, \tau_k) \} \prod_k \prod_{k-1} \cdots \prod_1 \big(1 + o(1)\big),$$

where

$$C(\tau_1,\ldots,\tau_k)=\sum_{j=2}^{k}\tau_j\sum_{\ell<j}h\Big(\frac{\log p_\ell}{\log p_j}\Big)$$

$$\prod\nolimits_j=\prod_{p_{j-1}<q<p_j^w}\left(1+\frac{\exp\big(i\tau_j h(\frac{\log q}{\log p_j})\big)-1}{q}\right)\quad(2\le j\le k)$$

and

$$\prod\nolimits_1=\prod_{q<p_1^w}\left(1+\frac{\exp\big(i\tau_1 h(\frac{\log q}{\log p_1})\big)-1}{q}\right).$$

To simplify the notation, we let

$$\kappa_\ell\overset{\text{def}}{=}\tau_\ell h\big(z/\xi(\wp_x)\big),\quad h_z(y)\overset{\text{def}}{=}\frac{h(y)}{h\big(z/\xi(\wp_x)\big)}.$$

The expressions $h_z\big(\frac{\log q}{\log p_j}\big)$ are small if $q<p_{j-1}^w$, and

$$\sum_{q<p_{j-1}^w}\frac{1}{q}h_z\Big(\frac{\log q}{\log p_j}\Big)\ll\frac{1}{h\big(z/\xi(\wp_x)\big)}\int_0^{we^{-1/\sqrt{w}}}\frac{h(u)}{u}\,du,\tag{7.13}$$

because of our choice of $R_x$ given by (7.12). Now (7.2) and (7.3) implies that the right hand side of (7.13) tends to 0 as $x\longrightarrow\infty$. Therefore we have, as $x\longrightarrow\infty$, that, setting $p_0=1$,

$$\prod\nolimits_j=\big(1+o(1)\big)\prod_{p_{j-1}<q<p_j^w}\left(1+\frac{\exp\big(i\kappa_j h_z(\frac{\log q}{\log p_j})\big)-1}{q}\right)\quad(j=1,\ldots,k),\tag{7.14}$$

and

$$\exp\big(iC(\tau_1,\ldots,\tau_k)\big)=1+o(1).\tag{7.15}$$

Estimations (7.14) and (7.15) are valid uniformly for $\kappa_1,\ldots,\kappa_k$ varying in an arbitrary bounded interval.

Because of (7.2), it follows that

$$\sum_{q<p_j^w}\frac{1}{q}h_z\Big(\frac{\log q}{\log p_j}\Big)\ll\frac{H(w)}{h(w)}\ll 1;$$

hence, repeating the argument used in the proof of Lemma 4, we get that

$$\prod\nolimits_j=\big(1+o(1)\big)\exp\Big(\int_0^w\frac{e^{i\kappa_j h_z(u)}-1}{u}\,du\Big)\quad(j=1,\ldots,k).$$

Let

$$B_{z,x}(\kappa)\overset{\text{def}}{=}\int_0^{z/\xi(\wp_x)}\frac{e^{i\kappa h_z(u)}-1}{u}\,du.$$

So far, we have proven that

$$A_{p_1,\dots,p_k}(\tau_1,\dots,\tau_k) = \big(1+o(1)\big)\frac{xw^k}{p_1\cdots p_k}\exp\Big(\sum_{j=1}^{k} B_{z,x}(\kappa_j)\Big).$$

Thus if we let

$$L_k \stackrel{\text{def}}{=} \sum_{p_1<\dots<p_k} A_{p_1,\dots,p_k}(\tau_1,\dots,\tau_k),$$

then we have

$$L_k = \big(1+o(1)\big)xw^k D_k \exp\Big(\sum_{j=1}^{k} B_{z,x}(\kappa_j)\Big), \tag{7.16}$$

with

$$D_k = \sum\nolimits^{\dagger} \frac{1}{p_1\cdots p_k},$$

where the † indicates that the sum runs over those $p_1 < \cdots < p_k$ ($p_j \in \wp_x'$, $j = 1,\dots,k$) for which there exist at least two primes $p_i < p_{i+1}$ such that $p_i > p_{i+1}^{1/R_x}$ with $R_x$ as in (7.12). We will prove that

$$\begin{aligned} D_k &= \frac{1}{k!}\Big(\sum_{p\in\wp_x'}\frac{1}{p}\Big)^k + O\Big(\big(\xi(\wp_x')\big)^k \log R_x\Big) \\ &= \frac{\big(\xi(\wp_x')\big)^k}{k!}\big(1+o(1)\big) = \frac{\big(\xi(\wp_x)\big)^k}{k!}\big(1+o(1)\big) \end{aligned} \tag{7.17}$$

which, substituted in (7.6), will yield

$$\frac{1}{x}L_k = z^k\frac{1+o(1)}{k!}\exp\Big(\sum_{j=1}^{k} B_{z,x}(\kappa_j)\Big).$$

To prove (7.17), we proceed as follows. Assume that $k$ is bounded by an arbitrary constant. Let $S_k = \sum^{\ddagger}\frac{1}{p_1\cdots p_k}$, where the ‡ indicates that the summation runs over all primes $p_1 < \cdots < p_k$ for which $p_j \in \wp_x'$ ($j = 1,\dots,k$). Then clearly $D_k \le S_k$. Observe that

$$S_k = \frac{1}{k!}\Big(\sum_{p\in\wp_x'}\frac{1}{p}\Big)^k + o\big(\xi(\wp_x)^k\big). \tag{7.18}$$

On the other hand,

$$\begin{aligned} S_k - D_k &\le \sum_{i=1}^{k-1}\sum_{\substack{p_1<\cdots<p_i<p_{i+1}<\cdots<p_k \\ p_{i+1}<p_i^{R_x}}} \frac{1}{p_1\cdots p_k} \\ &\le \sum_{i=1}^{k-1}\sum_{p_i<p_{i+1}<p_i^{R_x}}\frac{1}{p_{i+1}}\sum\frac{1}{p_1\cdots p_{i-1}p_ip_{i+2}\cdots p_k} \end{aligned} \tag{7.19}$$

$$< \log R_x \sum_{i=1}^{k-1} \sum \frac{1}{p_1 \cdots p_{i-1} p_i p_{i+2} \cdots p_k}$$

$$< \log R_x \frac{\left(\xi(\wp_x')\right)^{k-1}}{(k-1)!} = o\left(\xi(\wp_x')^k\right),$$

since, because of (7.12), $\log R_x = O(\sqrt{\xi(\wp_x)})$. The combination of (7.18) and (7.19) clearly yields (7.17).

Let $G_{z,x}(u)$ denote the distribution function which corresponds to the characteristic function $\exp\left(iB_{z,x}(\kappa)\right)$. Then, by the continuity theorem of the characteristic functions, we have, taking into account the asymptotic independency, that

$$\frac{1}{x} N_k(x) = \frac{\left(1+o(1)\right)}{k!} \left\{ \frac{G_{z,x}(1)}{z} \right\}^k .$$

Using the sieve formula, we conclude that

$$\frac{M_0(x)}{x} = \left(1+o(1)\right)\left\{1 - \frac{1}{1!}\frac{G_{z,x}(1)}{z} + \frac{1}{2!}\left(\frac{G_{z,x}(1)}{z}\right)^2 - \cdots\right\}$$
$$= \left(1+o(1)\right)e^{-\frac{G_{z,x}(1)}{z}}.$$

This last argumentation is correct, because $G_{z,x}(u)$ is continuous in $u$ and also continuous in the parameter $z$ as well and furthermore $N_1(x) - N_2(x) + \cdots + (-1)^{k-1}N_k(x)$ is an upper or lower estimate of $N'(x)$ according to the parity of $k$.

We have thus proven

THEOREM 9. *Let $h\colon [0,1) \longrightarrow \mathbf{R}$ be increasing in a neighbourhood of zero. Define $H(v) = \int_0^v \frac{h(u)}{u}\,du$ and assume that $h(v) \ll H(v) \ll h(v)$. Let $\wp_x$ be a set of primes such that* $\lim_{x\to\infty} \xi(\wp_x) = +\infty$. *Then the number of integers $n \le x$ for which (7.8) holds is*

$$x\left(1+o(1)\right)e^{-\frac{G_{z,x}(1)}{z}},$$

*where $G_{z,x}(u)$ is the distribution function of which the characteristic function is*

$$\exp\left\{\int_0^{z/\xi(\wp_x)} \frac{e^{i\kappa\frac{h(u)}{h(z/\xi(\wp_x))}} - 1}{u}\,du\right\}.$$

An interesting particular case is the following. Assume that $\lim_{v\to 0} \frac{h(\lambda v)}{h(v)} = t(\lambda)$ for every fixed $0 < \lambda \le 1$. Then, it is known (see Seneta [18]) that $t(\lambda) = \lambda^\alpha$ for some $\alpha > 0$, and since $t(\lambda)$ is increasing, then $h(v) = t(v)S(v)$, where $S(1/v)$ is a slowly oscillating function. For such a function $h$, we have that, as $x \longrightarrow \infty$,

$$B_{z,x}(\kappa) = \int_0^{z/\xi(\wp_x)} \frac{e^{i\kappa\frac{h(u)}{h(z/\xi(\wp_x))}} - 1}{u}\,du$$
$$= \int_0^{z/\xi(\wp_x)} \frac{e^{i\kappa\left(\frac{u}{z/\xi(\wp_x)}\right)^\alpha} - 1}{u}\,du + o(1)$$
$$= \int_0^1 \frac{e^{i\kappa v^\alpha} - 1}{v}\,dv + o(1).$$

From these observations, we deduce the following result.

THEOREM 10. *Assume that $h(u) = u^\alpha S(u)$ where $\alpha > 0$ and $S(1/u)$ is a slowly oscillating function. Let $G$ be the distribution function which corresponds to the characteristic function $\chi$ defined by*

$$\chi(\kappa) = \exp\left(\int_0^1 \frac{e^{i\kappa v^\alpha} - 1}{v}\, dv\right).$$

*Then, as $x \longrightarrow \infty$,*

$$\frac{1}{x}\#\{n \leq x : \Upsilon(n) \geq h(z/\xi(\wp_x))\} = (1 + o(1))e^{-G(1)/z},$$

*or similarly*

$$\frac{1}{x}\#\{n \leq x : (\xi(\wp_x))^\alpha \Upsilon(n) > z^\alpha\} = (1 + o(1))e^{-G(1)/z}.$$

PROOF. Apply Theorem 9 and replace $G_{x,z}(1)$ by $G(1)$.

REMARK. $\chi(\kappa)$ is in fact identical to the Fourier transform of the function $w_{1/\alpha}(u)$ introduced by Hensley [15]. Since Hensley gives an explicit definition of the $w$-functions as solutions of difference differential equations, the function $G$ can be explicitly defined.

## REFERENCES


1. J. D. Bovey, *On the size of prime factors of integers*, Acta. Arith. **33**(1977), 65–80.
2. N. G. de Bruijn, *On the number of uncancelled elements in the sieve of Eratosthenes*, Nederl. Akad. Wetensch., Proc. **53**, 803–812 = Indagationes Math. **12**(1950), 247–256.
3. ———, *On the number of positive integers $\leq x$ and free of prime factors $> y$*, Koninkl. Nederl. Akademie Van Wetenschappen, Series A **54**(1951), 49–60.
4. J. M. De Koninck, I. Kátai and A. Mercier, *Additive functions monotonic on the set of primes*, Acta Arith. **57**(1991), 41–68.
5. ———, *Continuity module of the distribution of additive functions related to the largest prime factors of integers*, Arch. Math. **55**(1990), 450–461.
6. J.M. De Koninck and J. Galambos, *The intermediate prime divisors of integers*, Proc. Amer. Math. Soc. **101**(1987), 213–216.
7. P. Erdős, *Some remarks about additive and multiplicative functions*, Bull. Amer. Math. Soc. **52**(1946), 527–537.
8. ———, *On some properties of prime factors of integers*, Nagoya Math. J. **27**(1966), 617–623.
9. A. S. Fainleib, *A generalization of Esseen's inequality and its application in probabilistic number theory*, Izvest. Akad. Nauk SSSR, Ser. Mat. **32** (1968), 859–879. English Transl. in Math. USSR Izvest. **2**(1968).
10. J. Galambos, *The sequences of prime divisors of integers*, Acta Arith. **31**(1976), 213–218.
11. ———, *On a problem of P. Erdős on large prime divisors of n and n + 1*, J. London Math. Soc. **13**(1976), 360–362.
12. ———, *Advanced Probability Theory*, Marcel Dekker, New York, Basel, 1988.
13. H. Halberstam and K.F. Roth, *Sequences*, Clarendon Press, Oxford, 1966.
14. G. H. Hardy and Ramanujan, *The total number of prime factors of a number n*, Quart. J. Math. (Oxford) **48**(1917), 76–92.
15. D. Hensley, *The convolution powers of the Dickman function*, J. London Math. Soc. (2) **33**(1986), 395–406.
16. J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, Translations of Mathematical Monographs, Vol. 11, AMS, Providence, R.I., 1964.
17. A. G. Postnikov, *Introduction to Analytic Number Theory*, AMS, 1968.

**18.** E. Seneta, *Regularly varying functions*, (LNM 508), Springer-Verlag, Berlin, Heidelberg, New York, 1976.
**19.** V. M. Zolotarev, *One–dimensional stable distributions*, Translations of Mathematical Monographs, AMS, Volume 65, Providence, Rhode Island, 1986.

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