ACTA ARITHMETICA
LVII (1991)

Additive functions monotonic on the set of primes

by

J. M. De Koninck* (Québec), I. Kátai** (Budapest)
and A. Mercier*** (Chicoutimi)

1. Introduction. Let $L\colon [1, \infty) \to [1, \infty)$ be a monotonically increasing function such that $\lim_{x\to\infty} L(x) = +\infty$. Let $f = f_L$ be a strongly additive function determined by $f(p) = L(p)$ on the set of primes. For an integer $n > 1$, let $p(n)$ and $P(n)$ denote the smallest and the largest prime factor of n, respectively.

De Koninck and Mercier [3] proved that if L is a slowly oscillating function which increases "fast enough" and $f = f_L$ is as above, then, as $x \to \infty$,

$$\sum_{2\leqslant n\leqslant x} f(P(n)) \sim \sum_{2\leqslant n\leqslant x} f(n).$$

In [2], we proved that, for a large class of strongly additive functions f_L, $\sum_{2\leqslant n\leqslant x} f_L(n)/L(P(n)) \sim x$ as $x \to \infty$.

Our purpose in this paper is to find necessary and sufficient conditions which L must satisfy in order that the functions

$$\frac{f_L(n)}{L(P(n))} \quad \text{or} \quad \frac{f_L(n)}{L(n)}$$

have mean values or limit distributions.

In what follows, $p, p_1, p_2, \ldots, q, q_1, q_2, \ldots$ stand for prime numbers. The letters $c, c_1, c_2, \ldots$ denote suitable positive constants (not necessarily the same at every occurrence) which may depend only on L. As usual, $\pi(x)$ denotes the number of primes up to x while $\pi(x, k, l)$ stands for the number of primes $p \leqslant x$, $p \equiv l \pmod{k}$. Finally $\varphi(n)$ denotes the Euler totient function, $\varphi_t(n)$ denotes the tth iterative function, $\varphi_0(n) = n$, $\varphi_1(n) = \varphi(n)$, $\varphi_t(n) = \varphi(\varphi_{t-1}(n))$.

---

* Research supported by grants of NSREC of Canada and FCAR of Québec.

** Work done while this author was a visiting professor at Temple University (Philadelphia). Research partially supported by the Hungarian Research Fund No. 907.

*** Research supported by a grant of NSERC of Canada.

2. Lemmata on primes and on additive functions.

2.1. Let $\Psi(x, y)$ be the number of integers n up to x for which $P(n) \leqslant y$. It is known (see de Bruijn [1]) that

$$\Psi(x, y) \leqslant x \exp\left(-c \frac{\log x}{\log y}\right) \tag{2.1}$$

is valid uniformly in x, $y \geqslant 2$, *and furthermore that*

$$\Psi(x, x^{\alpha}) = x(1+o_x(1))\varrho(1/\alpha) \quad (x \to \infty) \tag{2.2}$$

uniformly in every interval $\alpha \in [\delta, 1]$, $\delta > 0$, *where* ϱ *is a continuous function.*

2.2. By using elementary estimations on $\pi(x)$, one can obtain immediately that, if $s \geqslant 1$,

$$\sum_{p>H} \frac{(\log p)^{-s}}{p} \leqslant c \frac{1}{s(\log H)^s}, \tag{2.3}$$

$$\sum_{q<H} \frac{(\log q)^s}{q} \leqslant c \frac{(\log H)^s}{s} \tag{2.4}$$

uniformly in $H \geqslant 2$.

2.3. *Let* $c > 0$. *Then for* $0 < \delta < 1$,

$$\sum_{p \leqslant x^{\delta}} \frac{\exp\left(-c \frac{\log x}{\log p}\right)}{p} \leqslant e(\delta) + o_x(1) \tag{2.5}$$

where $e(\delta) \to 0$ *as* $\delta \to 0$.

Inequality (2.5) is an immediate consequence of the elementary estimate $\pi(x) < c_1 x/\log x$.

2.4. *If* $1 \leqslant k < x$ *and* $(k, l) = 1$, *we have*

$$\pi(x, k, l) < \frac{3x}{\varphi(k)\log(x/k)}. \tag{2.6}$$

For a proof of this result, see [5], Theorem 3.8.

2.5 (Theorem 2.3 in [5]). *Let* g *be a natural number,* a_i, b_i $(i = 1, \ldots, g)$ *be pairs of integers satisfying* $(a_i, b_i) = 1$ $(i = 1, \ldots, g)$, *and let*

$$E \stackrel{\text{def}}{=} \prod_{i=1}^{g} a_i \prod_{1 \leqslant r < s \leqslant g} (a_r b_s - a_s b_r) \neq 0.$$

Let y *and* x *be real numbers such that* $1 < y \leqslant x$. *Further, let* $\mathscr{B}$ *be a set of primes for which there exist constants* δ *and* A *such that*

$$\sum_{p<y;\, p \in \mathscr{B}} (1/p) \geqslant \delta \log\log y - A.$$

Then

$$\#\{n:\ n \in [x-y, x],\ (a_i n + b_i, \mathscr{B}) = 1,\ \forall i = 1, \ldots, g\}$$

$$\ll \prod_{p|E;\, p \in \mathscr{B}} \left(1 - \frac{1}{p}\right)^{w(p)-g} \frac{y}{(\log y)^{\delta g}},$$

where $w(p)$ *denotes the number of solutions of*

$$\prod_{i=1}^{g} (a_i n + b_i) \equiv 0 \pmod{p},$$

and where the constant implied by the $\ll$ *notation depends only on* g *and* A.

This theorem contains as a special case the following:

Corollary. *The number of solutions of the equation* $p-1 = aq$ *in prime variables* p, q, *where* p *runs in the range* $1 < p \leqslant x$, *is*

$$\ll \frac{x}{\varphi(a)\log^2(x/a)}$$

and the constant implied by the $\ll$ *symbol is absolute.*

2.6. *Let* $0 < \sigma < 1/4$, $U(x, \sigma)$ *be the number of those integers* n *up to* x, *the second largest prime factor* q *of which is larger than* $P(n)^{1-\sigma}$. *Then for each fixed* $\xi \in (0, 1/4)$,

$$U(x, \sigma) \leqslant \Psi(x, x^{\xi}) + x\left(\log \frac{1}{1-\sigma}\right)\left(\log \frac{1}{\xi}\right) + o(x). \tag{2.7}$$

This can be proven as follows. We separate the set of integers $n \leqslant x$ in two sets: those for which $P(n) < x^{\xi}$ and those for which $P(n) \geqslant x^{\xi}$. The first set has no more than $\Psi(x, x^{\xi})$ elements. To estimate the second one, we first fix p, then the second largest prime factor q of n is varying in $p^{1-\sigma} \leqslant q < p$. The size of this second set is therefore not larger than

$$x \sum_{x^{\xi} \leqslant p < x} \frac{1}{p} \sum_{p^{1-\sigma} \leqslant q < p} \frac{1}{q} = x\left(\log \frac{1}{\xi}\right)\left(\log \frac{1}{1-\sigma}\right) + o(x).$$

2.7. *Let* $g(n)$ *be an arbitrary strongly additive function. Then*

$$\sum_{n \leqslant x} \left(g(n) - \sum_{p \leqslant x} \frac{g(p)}{p}\right)^2 \leqslant cx \sum_{p \leqslant x} \frac{g^2(p)}{p}. \tag{2.8}$$

This is the so-called Turán–Kubilius inequality.

2.8. *Uniformly in* $y \geqslant 2$, $0 < \varepsilon < 1$, *we have*

$$\#\{p \leqslant y:\ P(p-1) \leqslant y^{\varepsilon}\} \leqslant c\lambda(\varepsilon)\, y/\log y,$$

where $\lambda(\varepsilon)$ *is a function tending to zero as* $\varepsilon \to 0$. (This inequality follows easily from Theorem 2.3 of [5] cited in 2.5.) *Consequently, if* $0 < \eta_2 < \eta_1$, *then*

$$\liminf_{x\to\infty} \sum_{x^{\eta_1}<q<x;\, P(q-1)<x^{\eta_2}} (1/q) \leqslant d(\eta_1, \eta_2),$$

where $\lim_{\eta_2\to 0} d(\eta_1, \eta_2) = 0$ *for every choice of* $\eta_1 > 0$.

3. Lemmata on functions L.

3.1. LEMMA 1. *Let* $L\colon [2, \infty) \to \mathbf{R}^+$ *be monotonic, continuous and satisfying* $\lim_{x\to\infty} L(x) = +\infty$. *Suppose that there exists a constant* $c > 0$ *such that*

$$\frac{1}{L(x)} \int_2^x \frac{L(u)}{u \log u}\, du \to c \quad \text{as } x \to \infty. \tag{3.1}$$

Then

$$L(x) = (\log x)^{1/c} H(x), \tag{3.2}$$

where $H(x)$ *is a very slowly oscillating function, i.e. a function such that*

$$\lim_{x_1\to\infty} \max_{x_1 \leqslant x_2 \leqslant x_1^2} \left| \frac{H(x_2)}{H(x_1)} - 1 \right| = 0. \tag{3.3}$$

Reciprocally, if L *satisfies* (3.2) *and* (3.3), *then* (3.1) *holds also.*

Remark 1. The notion of very slowly oscillating function was introduced by De Koninck and Mercier [3]. They defined such a function $H\colon [A, +\infty) \to \mathbf{R}^+$, $A > 0$, as one which satisfies

$$\lim_{x\to\infty} \frac{H(x^a)}{H(x)} = 1, \quad \text{for every fixed } a > 0.$$

This is clearly equivalent to our definition.

Proof of Lemma 1. Define

$$F(x) = \int_2^x \frac{L(u)}{u \log u}\, du. \tag{3.4}$$

Clearly

$$F'(x) = \frac{L(x)}{x \log x}. \tag{3.5}$$

By hypothesis, we have

$$L(x) \sim \frac{1}{c} F(x). \tag{3.6}$$

So, using (3.5) and (3.6), it follows that

$$\frac{F'(x)}{F(x)} \sim \frac{1}{cx \log x}.$$

Therefore

$$\log F(x) - \log F(2) = \int_2^x \frac{F'(t)}{F(t)}\, dt = \frac{1}{c} \int_2^x (1 + o(1)) \frac{dt}{t \log t} = \frac{1 + o(1)}{c} \log\log x,$$

whence we obtain

$$F(x) = (\log x)^{1/c} (\log x)^{o_x(1)}.$$

Thus, by (3.6),

$$L(x) = (\log x)^{1/c} (\log x)^{\delta(x)} = (\log x)^{1/c} H(x),$$

where $\delta(x) \to 0$ as $x \to \infty$.

Now we estimate $\log(H(x_2)/H(x_1))$ in the range $x_1 \leqslant x_2 \leqslant x_1^2$. First we observe that by (3.2) we have

$$\log \frac{H(x_2)}{H(x_1)} = \log \frac{L(x_2)}{L(x_1)} - \frac{1}{c} \log \frac{\log x_2}{\log x_1}.$$

Using (3.6), we then obtain

$$\log \frac{L(x_2)}{L(x_1)} = \log \frac{F(x_2)}{F(x_1)} + o_{x_1}(1).$$

On the other hand, we have

$$\log \frac{F(x_2)}{F(x_1)} = \int_{x_1}^{x_2} \frac{1 + o_t(1)}{ct \log t}\, dt = \frac{1}{c} \log \frac{\log x_2}{\log x_1} + \int_{x_1}^{x_2} \frac{o_t(1)}{ct \log t}\, dt.$$

Combining these last two estimates, we conclude that

$$\log \frac{H(x_2)}{H(x_1)} = \int_{x_1}^{x_2} \frac{o_t(1)}{ct \log t}\, dt,$$

which implies (3.3).

We now prove the second assertion. For this we assume that (3.2) and (3.3) are satisfied. So we have, for every fixed $\delta > 0$,

$$\int_{x^\delta}^x \frac{L(u)}{L(x)} \frac{1}{u \log u}\, du = (1 + o_x(1)) \int_{x^\delta}^x \left(\frac{\log u}{\log x} \right)^{1/c} \frac{du}{u \log u} \tag{3.7}$$

$$= (1 + o_x(1))\, c\, (1 - \delta^{1/c}).$$

This implies that, uniformly in x,

$$\int_{x^{1/2}}^x \frac{L(u)}{u \log u}\, du < c_1 L(x).$$

Consequently

$$\int_2^x \frac{L(u)}{u\log u}\,du < c_1\left(L(x)+L(x^{1/2})+L(x^{1/2^2})+\ldots\right)$$

$$= c_1 L(x)\left(1+\frac{L(x^{1/2})}{L(x)}+\frac{L(x^{1/2^2})}{L(x^{1/2})}\frac{L(x^{1/2})}{L(x)}+\ldots\right).$$

But from (3.2) it is clear that, uniformly in $x > c_2$, there exists $0 < \theta < 1$ such that

$$L(x^{1/2})/L(x) < \theta.$$

Thus we have

$$\int_2^x \frac{L(u)}{u\log u}\,du < c_2 L(x). \tag{3.8}$$

Then the conclusion follows easily from (3.7) and (3.8).

Remark 2. *Assume that the conditions of Lemma 1 are satisfied with $c = 0$. Then*

$$\frac{\log L(x)}{\log\log x} \to \infty \quad \textit{as } x\to\infty. \tag{3.9}$$

This result can be obtained easily if we observe that (3.1) implies that for every $\varepsilon > 0$ we have $L(x^{1/2})/L(x) < \varepsilon$ whenever $x > x_0(\varepsilon)$.

Remark 3. *Assume that the conditions of Lemma 1 are satisfied with $c = \infty$. Then*

$$\frac{\log L(x)}{\log\log x} \to 0 \quad \textit{as } x\to\infty. \tag{3.10}$$

In order to prove this, first let F be as in (3.4). Then $L(x) = o_x(1)\,F(x)$, and using (3.5) we deduce that

$$\frac{F'(u)}{F(u)} = o_u(1)\frac{1}{u\log u} \quad (u\to\infty). \tag{3.11}$$

This implies that

$$\log F(x) = \log F(2) + \int_2^x o_u(1)\frac{du}{u\log u} = \delta(x)\log\log x, \tag{3.12}$$

where $\delta(x)\to 0$ as $x\to\infty$. But since $L(x) \ll F(x)$, (3.10) follows from (3.12).

3.2. In this section we assume that $L\colon [2,\infty)\to \boldsymbol{R}^+$ is a monotonic function satisfying $\lim_{x\to\infty} L(x) = +\infty$. Further, for each $\Delta\in(0,1)$, let $(\mathscr{H}_\Delta)$ denote the condition:

$$(\mathscr{H}_\Delta)\colon \qquad \lim_{x\to\infty}\frac{L(x^{1-\Delta})}{L(x)} = 0.$$

Remark 4. A function L which satisfies condition $(\mathscr{H}_\Delta)$ for some $\Delta\in(0,1)$ and which is also a slowly oscillating function (that is, such that $\lim_{x\to\infty} L(ax)/L(x) = 1$ for each fixed $a > 0$) must satisfy

$$\frac{L'(x)}{L(x)}\,x\log x\to\infty \quad \text{as } x\to\infty,$$

which means essentially that it increases faster than any power of $\log x$.

LEMMA 2. *Assume that condition $\mathscr{H}_{1/2}$ holds. Then there exists a monotonic function H such that $\lim_{y\to\infty} H(y) = +\infty$ and such that*

$$\frac{L(y)}{L(x)} < \left(\frac{\log y}{\log x}\right)^{H(y)} \quad \textit{whenever } 2\leqslant y\leqslant x^{1/16}. \tag{3.13}$$

Proof. First we let

$$\varepsilon(y) \stackrel{\text{def}}{=} \max_{y\leqslant z\leqslant\sqrt{x}} \frac{L(z)}{L(x)},$$

and define $H(y)$ by

$$\varepsilon(y) = 2^{-2H(y)}.$$

Since clearly $\lim_{y\to\infty}\varepsilon(y) = 0$, we have $\lim_{y\to\infty} H(y) = +\infty$. Let y and x be any pair of real numbers satisfying the condition $2\leqslant y\leqslant x^{1/16}$. Further, let k be the largest integer for which $y^{2^k}\leqslant\sqrt{x}$. Clearly $k\geqslant 1$. Since ε is monotonic, we can write

$$\frac{L(y)}{L(x)} \leqslant \frac{L(y)}{L(y^2)}\cdots\frac{L(y^{2^{k-1}})}{L(y^{2^k})} \leqslant \varepsilon(y)^k = 2^{-2kH(y)};$$

further, since

$$y^{2^{k+1}} > \sqrt{x}, \qquad 2^{k+1}\geqslant\frac{1}{2}\frac{\log x}{\log y}, \qquad \text{i.e. } \frac{1}{2^k}\leqslant\frac{4\log y}{\log x},$$

we have

$$\frac{L(y)}{L(x)} \leqslant \left(\frac{4\log y}{\log x}\right)^{2H(y)}.$$

Finally, since $H(y)\geqslant 0$ and

$$\left(\frac{4\log y}{\log x}\right)^2 \leqslant \frac{\log y}{\log x},$$

we easily obtain (3.13).

LEMMA 3. *Assume that $(\mathscr{H}_\Delta)$ holds for every Δ in $(0, 1)$. Let C and ε be arbitrary positive numbers. Then there exists a constant $x_0 = x_0(C,\varepsilon)$ such that*

(3.14) $$\frac{L(y)}{L(x)} \leqslant \left(\frac{\log y}{\log x}\right)^C \quad \textit{whenever } x_0 < y < x^{1-\varepsilon}.$$

Proof. The proof is similar to the one of Lemma 2. If $y < x^{1/16}$, then (3.14) is a consequence of (3.13). So let us take $y \geqslant x^{1/16}$. Set $\sigma = 1/(1-\varepsilon)$ and, for each $k = 1, 2, \ldots$, define $y_k = y^{\sigma^k}$. Further, let T be defined by $y_T \leqslant x^{1-\varepsilon} < y_{T+1}$ and let A be a large positive number such that

(3.15) $$e^{AT} > (1-\varepsilon)^{-C}\sigma^{TC}.$$

Clearly (3.15) implies that

$$(\log y/\log x)^C \geqslant e^{-AT} \quad \text{if } y \leqslant x^{1-\varepsilon}.$$

Choose y_0 large enough so that

$$\max_{z \geqslant y_0} \frac{L(z)}{L(z^\sigma)} \leqslant e^{-A}.$$

Then, if $y \geqslant y_0$, we have

(3.16) $$\frac{L(y)}{L(x)} \leqslant \frac{L(y)}{L(y_1)} \cdots \frac{L(y_T)}{L(y_{T+1})} \leqslant e^{-AT}.$$

Observing that

$$\log y_T = \sigma^T \log y,$$
$$\log y_T \leqslant (1-\varepsilon)\log x,$$
$$(1-\varepsilon)\log x < \log y_{T+1} < \sigma \log y_T,$$

(3.14) follows rapidly.

4. On additive functions satisfying $f(n)/f(P(n)) \to 1$ for almost all n. Assume that L: $[2, \infty) \to \mathbf{R}^+$ is a monotonic function satisfying $\lim_{x\to\infty} L(x) = +\infty$ and that $f = f_L$ is a strongly additive function. Let

(4.1) $$u(n) \overset{\text{def}}{=} \frac{f(n)}{f(P(n))} - 1 = \frac{1}{L(P(n))} \sum_{q|n;\, q<P(n)} L(q).$$

THEOREM 1. *Let L, f, u be as above. Assume that $u(n) \to 0$ as $n \to \infty$ for almost all n. Then $(\mathscr{H}_\Delta)$ holds for every $0 < \Delta < 1$. On the other hand, if $(\mathscr{H}_\Delta)$ holds for every $0 < \Delta < 1$, then, for every $a > 0$,*

(4.2) $$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leqslant x} (e^{au(n)} - 1) = 0,$$

in which case $u(n) \to 0$ for almost all n.

Proof. I. First assume that $u(n) \to 0$ for almost all n and suppose that $(\mathscr{H}_\delta)$ fails to hold for some $\delta \in (0, 1)$, which we can assume to be smaller than 1/2. We shall show that this leads to a contradiction.

Let Δ be defined by $\delta = 3\Delta/(1+\Delta)$. Let $1 \leqslant x_1 < x_2 < \ldots$ be a sequence of real numbers such that

$$L(x_\nu^{1-2\Delta}) > cL(x_\nu^{1+\Delta}) \quad (\nu = 1, 2, \ldots)$$

for some $c > 0$. Define $Y_\nu = x_\nu^{3-4\Delta}$. Now let p and q run over the intervals

$$x_\nu^{1-2\Delta} < q < x_\nu^{1-\Delta}, \quad x_\nu < p < x_\nu^{1+\Delta}.$$

If $n \leqslant Y_\nu$ and n is a multiple of pq, then $n = mpq$, where

$$m \leqslant Y_\nu/(pq) \leqslant x_\nu^{1-2\Delta} \quad (< q),$$

$P(n) = p$ and q is the second largest prime factor of n. With this it is clear that

$$u(n) \geqslant \frac{L(q)}{L(p)} \geqslant \frac{L(x_\nu^{1-2\Delta})}{L(x_\nu^{1+\Delta})} > c.$$

We now count the number S of such integers n. Clearly

$$S = \sum_{p,q} \left[\frac{Y_\nu}{pq}\right] = Y_\nu \left(\sum \frac{1}{q}\right)\left(\sum \frac{1}{p}\right) + o(Y_\nu).$$

Since

$$\sum \frac{1}{q} = \log \frac{1-\Delta}{1-2\Delta} + o_\nu(1) \quad \text{and} \quad \sum \frac{1}{p} = \log(1+\Delta) + o_\nu(1),$$

we may conclude that

$$S \geqslant dY_\nu + o(Y_\nu), \quad \text{with} \quad d = \log\frac{1-\Delta}{1-2\Delta} \cdot \log(1+\Delta) > 0.$$

This contradicts our assumptions. The first assertion is thus proven.

II. Assume now that $(\mathscr{H}_\Delta)$ holds for every $\Delta \in (0, 1)$. Let

$$S(x, u, a) \overset{\text{def}}{=} \sum_{2 \leqslant n \leqslant x} (e^{au(n)} - 1).$$

We want to show that

(4.3) $$S(x, u, a) = o(x).$$

Since $u(n) \leqslant \omega(n)$ and

$$\sum_{2\leqslant n\leqslant x} e^{2a\omega(n)} \ll x(\log x)^{e^{2a}-1},$$

a simple application of the Cauchy–Schwarz inequality shows that

$$\sum_{n\in\mathcal{J}_x} (e^{au(n)}-1) = o(x),$$

where $\mathcal{J}_x$ is a subset of $\{n\in N:\ n\leqslant x\}$ for which

$$\operatorname{card}(\mathcal{J}_x) = O\left(x/(\log x)^{e^{2a}}\right).$$

Therefore, because of (2.1), we may omit those integers $n\leqslant x$ for which $P(n) < \exp\left((\log x)^{1/2}\right)$.

Let $u_1(n)$, $u_2(n)$ be arbitrary positive quantities such that $u_1(n)+u_2(n) = u(n)$. First consider the obvious identity

$$e^{au(n)}-1 = (e^{au_1(n)}-1)(e^{au_2(n)}-1)+(e^{au_1(n)}-1)+(e^{au_2(n)}-1).$$

Now since

$$(e^{au_j(n)}-1)^2 \leqslant e^{2au_j(n)}-1 \qquad (j=1,2),$$

we easily obtain, using the Cauchy–Schwarz inequality,

(4.4) $S(x, u, a)$

$$\leqslant S(x, u_1, a)+S(x, u_2, a)+\left(S(x, u_1, 2a)\right)^{1/2}\left(S(x, u_2, 2a)\right)^{1/2}.$$

We now choose small positive numbers ε, η arbitrarily and pick according to these a large positive C such that the inequality

$$(1-\varepsilon)^{C-1} < \eta \tag{4.5}$$

is satisfied. Let x_0 be the constant involved in (3.14). We let

$$u_1(n) \overset{\text{def}}{=} \sum_{q\mid n,\, q\leqslant x_0} \frac{L(q)}{L(P(n))}, \qquad u_2(n) \overset{\text{def}}{=} u(n)-u_1(n).$$

Since we have assumed that $P(n)\geqslant \exp(\sqrt{\log x})$, it is clear that, because of (3.14), one has $u_1(n)\to 0$ and hence $S(x, u_1, a) = o(x)$ as $x\to\infty$. Therefore, keeping in mind (4.4), in order to obtain (4.3), it is enough to prove that $S(x, u_2, 2a) = o(x)$ as $x\to\infty$. To do this, we split u_2 into two parts, $u_2 = u_3+u_4$, where

$$u_3(n) \overset{\text{def}}{=} \sum_{q\mid n,\, x_0<q<P(n)^{1-\varepsilon}} \frac{L(q)}{L(P(n))}, \qquad u_4(n) \overset{\text{def}}{=} \sum_{q\mid n;\, P(n)^{1-\varepsilon}\leqslant q<P(n)} \frac{L(q)}{L(P(n))}.$$

We shall show that

$$(1/x)\, S(x, u_i, 4a) < \delta \qquad (i=3,4)$$

for every $\delta>0$ and every large x, thus establishing the proof of (4.3) and hence of Theorem 1.

We first estimate $S(x, u_3, 4a)$. Using (4.5) and Lemma 3, we obtain

$$u_3(n) \leqslant \frac{\log x}{\log P(n)}(1-\varepsilon)^{C-1} < \eta\frac{\log x}{\log P(n)}.$$

Thus

$$S(x, u_3, 4a) \leqslant \sum_{\exp\sqrt{\log x}\leqslant p\leqslant x} \left(e^{4a\eta\frac{\log x}{\log p}}-1\right)\Psi(x/p, p).$$

Using (2.1), we may write

(4.6) $S(x, u_3, 4a)$

$$\leqslant x \sum_{\exp\sqrt{\log x}\leqslant p<x^{1/4}} \frac{1}{p}\left(e^{4a\eta\frac{\log x}{\log p}}-1\right)e^{-\frac{c}{2}\frac{\log x}{\log p}} + x\sum_{x^{1/4}<p\leqslant x}\frac{1}{p}(e^{16a\eta}-1).$$

The second sum is bounded by $2(e^{16a\eta}-1)$. To estimate the first sum, we can separate it into two parts, namely the one for which $p<x^{8a\eta}$ and the one for which $p\geqslant x^{8a\eta}$. Thus we obtain that the first sum on the right of (4.6) is

$$\ll \sum_{\exp\sqrt{\log x}\leqslant p<x^{8a\eta}} \frac{1}{p}e^{-\frac{c}{4}\frac{\log x}{\log p}} + 8a\eta\sum_{p<x^{1/4}}\frac{\log x}{p\log p}e^{-\frac{c}{2}\frac{\log x}{\log p}}.$$

The second sum in the above expression is clearly bounded. On the other hand, one can see, using (2.5), that the first sum above is bounded by a function of η which tends to 0 as $\eta\to 0$. Hence we have proven that $S(x, u_3, 4a) < (\delta/2)x$ if η is small enough and x large enough.

We now proceed to estimate $S(x, u_4, 4a)$. Let $\varepsilon>0$ be small but fixed. For each integer n ($\geqslant \exp\sqrt{\log x}$), we let $P(n) = p$, and $q_1, \ldots, q_r$ be all its prime divisors which belong to the interval $[p^{1-\varepsilon}, p]$. If there are no such primes q_j, then $u_4(n) = 0$, and in general, $u_4(n)\leqslant r$. Therefore

$$S(x, u_4, 4a) \leqslant \sum_{r=1}^{\infty} (e^{4ar}-1)\, T_r(x), \tag{4.7}$$

where $T_r(x)$ stands for the number of those integers $n\leqslant x$ such that $P(n) = p > \exp\sqrt{\log x}$ and which have exactly r prime divisors in $[p^{1-\varepsilon}, p]$. We will show that there exists an absolute constant $c_1>0$ such that

$$T_r(x) \leqslant c_1 x(2\varepsilon)^r/r!. \tag{4.8}$$

Setting this in (4.7), it will follow that

$$S(x, u_4, 4a) \leqslant c_1 x\sum_{r=1}^{\infty} e^{4ar}(2\varepsilon)^r/r! = c_1 x(e^{2\varepsilon e^{4a}}-1).$$

Hence, if ε is sufficiently small, we will have

$$S(x, u_4, 4a) \leqslant c_1 x\, 4e^{4a}\varepsilon,$$

from which it follows that $S(x, u_4, 4a) = o(x)$. This will end the proof of Theorem 1 if we can prove (4.8). For this, we first write

$$T_r(x) = \sum_{\exp\sqrt{\log x} \leqslant p \leqslant x} \sum_{pq_1 \dots q_r m \leqslant x} 1,$$

where in the inner sum the q_i's and m satisfy

$$p^{1-\varepsilon} \leqslant q_1 < \dots < q_r \leqslant p, \quad P(m) \leqslant p^{1-\varepsilon}.$$

Hence we have

$$(4.9) \qquad T_r(x) = \sum_{\exp\sqrt{\log x} \leqslant p \leqslant x} \sum_{p^{1-\varepsilon} \leqslant q_1 < \dots < q_r < p} \Psi\left(\frac{x}{pq_1 \dots q_r}, p^{1-\varepsilon}\right).$$

We now separate the sum over p into two sums, according as $p > x^{1/(8r)}$ or $p \leqslant x^{1/(8r)}$. We first consider the one with $p > x^{1/(8r)}$. In this case, since $p^{r(1-\varepsilon)+1} < x$, it follows that $p < x^{1/r}$ and therefore the part of the sum in (4.9) concerned by those p's is clearly

$$\ll x \sum_{x^{1/(8r)} < p < x^{1/r}} \frac{1}{p} \frac{1}{r!} \left(\sum_{p^{1-\varepsilon} < q < p} \frac{1}{q}\right)^r \leqslant 3x \frac{(2\varepsilon)^r}{r!}.$$

If $p \leqslant x^{1/(8r)}$, then $pq_1 \dots q_r < p^{r+1} < x^{1/4}$ and

$$\Psi\left(\frac{x}{pq_1 \dots q_r}, p^{1-\varepsilon}\right) \leqslant \frac{x}{pq_1 \dots q_r} \exp\left(-\frac{c}{2} \frac{\log x}{\log p}\right).$$

Summing first on $q_1, \dots, q_r$, the sum with $p \leqslant x^{1/(8r)}$ is bounded by

$$\frac{(2\varepsilon)^r}{r!} \sum_p \frac{1}{p} \exp\left(-\frac{c}{2} \frac{\log x}{\log p}\right) \ll 1.$$

This proves (4.8) and finishes the proof of Theorem 1.

Remark 5. Let $P_1 > P_2 > \dots > P_k$ be the largest prime factors of n and define

$$u_k(n) = \frac{1}{L(P_k)} \sum_{q|n, q < P_k} L(q).$$

By the method used in the proof of Theorem 1, one could prove the following generalization of Theorem 1:

THEOREM 1′. *Assume that condition* $(\mathscr{H}_\Delta)$ *holds for every* $0 < \Delta < 1$. *Then, for every positive integer* k *and every* $a > 0$,

$$\frac{1}{x} \sum_{n \leqslant x} (e^{au_k(n)} - 1) \to 0.$$

5. The behaviour of $f(n)/f(P(n))$ on subsets of N. In this section we shall show that the sufficiency of the conditions (in Theorem 1) remains true if we replace the set of all integers by some subset satisfying certain conditions. Hence we state:

THEOREM 2. *Let* $\mathscr{A} = \{a_1, a_2, \dots\}$ *be an infinite sequence of not necessarily distinct integers such that* $a_n = O(n^k)$, *where* k *is a fixed positive real number. Assume that*

$$(5.1) \qquad \limsup_{x \to \infty} \frac{1}{x} \# \{n \leqslant x \colon P(a_n) < x^\delta\} \leqslant c(\delta),$$

where

$$(5.2) \qquad \lim_{\delta \to 0} c(\delta) = 0.$$

Further, for $0 < \varepsilon < \delta$, *let* $S(x, \delta, \varepsilon)$ *be the number of those integers* $n \leqslant x$ *for which there exists a suitable couple of primes* q_1, q_2 *such that*

$$q_1 q_2 | a_n; \quad x^\delta < q_1 < q_2 < q_1 x^\varepsilon$$

and set

$$(5.3) \qquad d(\varepsilon, \delta) \stackrel{\text{def}}{=} \limsup_{x \to \infty} \frac{1}{x} S(x, \delta, \varepsilon).$$

Assume that

$$(5.4) \qquad \lim_{\varepsilon \to 0} d(\varepsilon, \delta) = 0 \quad \text{for every } 0 < \delta < 1/2$$

and furthermore that for L *the condition* $(\mathscr{H}_\Delta)$ *is valid for every* $0 < \Delta < 1$.
Then $u(a_n) \to 0$ *for almost all* n.

Proof. First we choose $0 < \varepsilon < \delta$ fixed. We then define the set $\mathscr{B} = \mathscr{B}_{x,\delta,\varepsilon}$ as the set of all integers $n \leqslant x$ such that $P(a_n) > x^\delta$ and for which there exist no pairs q_1, q_2 of primes satisfying the conditions

$$q_1, q_2 \in [x^{\delta/2}, P(a_n)], \quad q_1 q_2 | a_n, \quad q_1 < q_2 < q_1 x^\varepsilon.$$

Let $\bar{\mathscr{B}}$ be the complementary set, that is,

$$\bar{\mathscr{B}} = \{1, 2, \dots, [x]\} \backslash \mathscr{B}.$$

We shall first prove that $u(a_n)$ is small for every $n \in \mathscr{B}$. So let $a_n = P(a_n)^\beta b_n d_n$, where $P(a_n)^\beta \| a_n$, b_n is composed of the prime power divisors q^α of a_n satisfying $x^{\delta/2} < q < P(a_n)$, and d_n of those for which $q \leqslant x^{\delta/2}$. Let also

$$\sigma_1(n) \stackrel{\text{def}}{=} \sum_{q|b_n} \frac{L(q)}{L(P(a_n))}, \qquad \sigma_2(n) \stackrel{\text{def}}{=} \sum_{q|d_n} \frac{L(q)}{L(P(a_n))}.$$

It is clear that

$$u(a_n) \leqslant \sigma_1(n) + \sigma_2(n).$$

Let $n \in \mathscr{B}$. Since $q|b_n$ implies that $q < P(a_n) x^{-\varepsilon}$, by Lemma 3, we have

$$\sigma_1(n) \leqslant \sum_{q|b_n} \left(\frac{\log q}{\log P(a_n)}\right)^C < \left(1-\frac{\varepsilon}{k}\right)^C \sum_{q|b_n} 1.$$

Since $b_n \leqslant a_n \leqslant Cn^k$, it follows that $\sum_{q|b_n} 1 \leqslant 2k/\delta$, if x is large. Hence we have

$$\sigma_1(n) \leqslant \frac{2k}{\delta}\left(1-\frac{\varepsilon}{k}\right)^C. \tag{5.5}$$

Now define

$$I_h \stackrel{\text{def}}{=} [x^{\delta/2^{h+1}}, x^{\delta/2^h}], \qquad \tau_h(n) \stackrel{\text{def}}{=} \frac{1}{L(P(a_n))} \sum_{q\in I_h;\, q|d_n} L(q).$$

Then

$$\sigma_2(n) \leqslant \sum_{h=1}^{\infty} \tau_h(n). \tag{5.6}$$

Denote by $q_1, q_2, \ldots, q_R$ all the prime divisors of a_n in I_h. Since $q_i \geqslant x^{\delta/2^{h+1}}$, $\prod q_i \leqslant a_n \leqslant Cx^k$, it follows that $R \leqslant 2k \cdot \frac{2^{h+1}}{\delta}$. On the other hand, since $L(q)/L(P(a_n)) \leqslant 2^{-hC}$ if $q > x_0(C, 1/2)$, we have

$$\tau_h(n) \leqslant 2k \frac{2^{h+1}}{\delta} \cdot 2^{-hC}. \tag{5.7}$$

Choose C large. It is clear that

$$\max_{n\leqslant x;\, P(a_n) > x^\delta} \left(\frac{1}{L(P(a_n))} \sum_{q|a_n;\, q\leqslant x_0(C,1/2)} L(q)\right) \to 0 \quad \text{as } x\to\infty.$$

Hence we have

$$\sigma_2(n) \leqslant \frac{16k\cdot 2^{-C}}{\delta} + o_x(1)$$

if $C \geqslant 3$. For each fixed $\xi > 0$, let

$$T_x \stackrel{\text{def}}{=} \frac{1}{x} \# \{n \leqslant x:\ u(a_n) > \xi\}.$$

We shall prove that $\lim_{x\to\infty} T_x = 0$.

Let δ be an arbitrary small number. Then it is easy to see that we can pick ε, $0 < \varepsilon < \delta$, small enough, and $C > 0$ large enough, so that

$$\frac{2}{\delta}\left(1-\frac{\varepsilon}{k}\right)^C + \frac{16k\cdot 2^{-C}}{\delta} < \frac{\xi}{2}.$$

Hence we may conclude that $u(a_n) < \xi$ for each $n \in \mathscr{B}_{x,\delta,\varepsilon}$, whenever x is sufficiently large.

Finally, we estimate card $(\mathscr{B})$. If $n \in \mathscr{B}$, then either $P(a_n) \leqslant x^\delta$, or there exist q_1, q_2 such that $q_1 q_2 | a_n$, $x^{\delta/2} \leqslant q_1 < q_2 < q_1 x^\varepsilon$. So, by our assumptions (5.3) and (5.4), we have

$$\limsup T_x \leqslant c(\delta/2) + d(\varepsilon, \delta/2).$$

We first let ε tend to 0, and then we let δ tend to 0. It follows that $\limsup T_x = 0$. Since this is true for every $\xi > 0$, our theorem follows.

Remark 6. One can easily check that the conditions hold for $\mathscr{A} = N$.

6. Iterations of the totient function.

THEOREM 3. *Let $\mathscr{A} = \{\varphi_t(n):\ n = 1, 2, \ldots\}$, where $\varphi_t(n)$ is the t-fold iterate of the Euler totient function. Then the conditions of Theorem 2 are satisfied. Consequently, if $(\mathscr{H}_\Delta)$ is satisfied for every $0 < \Delta < 1$, then*

$$u(\varphi_t(n)) \to 0 \quad (n \to \infty)$$

for almost all n.

Proof. Let $R_0(n) = n$, $R_t(n) = \varphi_t(n) R_{t-1}(n)$, i.e. $R_t(n) = n\varphi(n)\ldots\varphi_t(n)$. In order to prove our claim, we only need to prove that the conditions of Theorem 2 hold for

$$\mathscr{A}_t^* \stackrel{\text{def}}{=} \{R_t(n):\ n = 1, 2, \ldots\}.$$

We proceed by induction on t. If $t = 0$, then $\mathscr{A}_0^* = N$, and consequently the conditions of Theorem 2 hold. Assume now that $t \geqslant 1$. We shall proceed in three major steps:

1. Let $\mathscr{B}_t = \{p_1, p_2, \ldots, p_R\}$ be an arbitrary set of primes $p_i \leqslant x$. Define $\mathscr{B}_{t-1}, \ldots, \mathscr{B}_0$ as follows. $\mathscr{B}_{t-1}$ is the set of those primes $q \leqslant x$ for which there exists $p \in \mathscr{B}_t$ such that $p | q-1$. If $\mathscr{B}_{t-1}, \ldots, \mathscr{B}_{j+1}$ are defined, then $\mathscr{B}_j$ is the set of those primes $q \leqslant x$ for which there exists $p \in \mathscr{B}_{j+1}$ such that $p | q-1$.

For an arbitrary subset $\mathscr{D}$ of primes, let

$$A_t(x|\mathscr{D}) \stackrel{\text{def}}{=} \# \{n \leqslant x:\ (R_t(n), \mathscr{D}) > 1\}. \tag{6.1}$$

Here $(R_t(n), \mathscr{D}) > 1$ means that there exists $q \in \mathscr{D}$ for which $q | R_t(n)$. Hence if we let

$$E_t(x|\mathscr{D}) \stackrel{\text{def}}{=} \# \{n \leqslant x:\ (R_t(n), \mathscr{D}) > 1 \text{ and } (R_{t-1}(n), \mathscr{D}) = 1\},$$

it follows that

$$A_t(x|\mathscr{D}) = \sum_{j=0}^{t} E_j(x|\mathscr{D}). \tag{6.2}$$

Letting

$$s(\mathscr{D}) \stackrel{\text{def}}{=} \sum_{p\in\mathscr{D}} 1/p, \tag{6.3}$$

we clearly have

(6.4) $$E_0(x|\mathscr{D}) \leqslant xs(\mathscr{D}),$$

for every choice of $\mathscr{D}$. Furthermore, if $\mathscr{D} = \mathscr{B}_t$ and n is counted in $E_t(x|\mathscr{D})$, then n is counted in $E_{t-1}(x|\mathscr{B}_{t-1})$ as well. Indeed, let $\varphi_{t-1}(n) = r_1^{\alpha_1}\cdot\ldots\cdot r_l^{\alpha_l}$ and assume that $(R_{t-1}(n), \mathscr{B}_t) = 1$, $(R_t(n), \mathscr{B}_t) > 1$. Since there exists $p|\varphi_t(n) = \prod_{i=1}^{l} r_i^{\alpha_i-1}(r_i-1)$, $p \in \mathscr{B}_t$ and $(p, R_{t-1}(n)) = 1$, it follows that $r_i - 1 \equiv 0 \pmod p$ for some $r_i \in \mathscr{B}_{t-1}$. So we have

(6.5) $$E_t(x|\mathscr{B}_t) \leqslant E_{t-1}(x|\mathscr{B}_{t-1}).$$

Now let δ be a small positive number and assume that all the elements of $\mathscr{B}_t$ are larger than x^δ. Then the elements of $\mathscr{B}_j$ are larger than x^δ for $0 \leqslant j < t$, as well.

Further, let η be a small positive number that may depend on δ. We shall choose it later explicitly in such a way that it will allow us to estimate $s(\mathscr{B}_j)$ from $s(\mathscr{B}_{j+1})$.

First we observe that

$$s(\mathscr{B}_j) \leqslant \sum_{p\in\mathscr{B}_{j+1}} \sum_{q\equiv 1 \pmod p} 1/q = \sum_{p\in\mathscr{B}_{j+1}} (U_1(p)+U_2(p)),$$

where in $U_1(p)$ we sum over those q for which $q < p^{1+\eta}$, while in $U_2(p)$ we sum over those q such that $q > p^{1+\eta}$.

Let $M_0 = p^{1+\eta}$, $M_\nu = e^\nu M_0$, $\nu = 1, 2, \ldots$ Starting from the inequality

$$\pi(2M_\nu, p, 1) < \frac{cM_\nu}{p\log(M_\nu/p)}$$

valid with an absolute constant c (see (2.6)), we obtain

$$U_2(p) = \sum_{M_\nu\leqslant x} \sum_{q\in[M_\nu, M_{\nu+1}]} \frac{1}{q} < \frac{c}{p}\sum_\nu \frac{1}{\nu+\eta\log p},$$

where ν runs over the nonnegative integers satisfying $M_\nu \leqslant x$. Hence we have

$$U_2(p) \leqslant \frac{c}{p}\log\frac{1}{\eta\delta};$$

therefore

$$\sum_{p\in\mathscr{B}_{j+1}} U_2(p) \leqslant As(\mathscr{B}_{j+1}),$$

where

(6.6) $$A = c\log(1/(\eta\delta)).$$

We now proceed to estimate

(6.7) $$\sum_{p\in\mathscr{B}_{j+1}} U_1(p).$$

If p, q are two primes occurring in (6.7), then $q-1 = ap$, where $1 \leqslant a \leqslant p^\eta$. Therefore the sum (6.7) is not greater than $\sum_{p,q} 1/q$, where the summation is extended over all solutions of the equation $q-1 = ap$, $1 \leqslant a \leqslant p^\eta$ in primes p, q satisfying $x^\eta \leqslant p < q \leqslant x$. To estimate it, we shall use the Corollary stated in Section 2.5. We split the interval $[x^\eta, x]$ in intervals of the type $[M_\nu, M_{\nu+1}]$, where $M_0 = x^\delta$, $M_\nu = 2^\nu M_0$. Since for every fixed a, the number of solutions of $p-1 = aq$ where $q \in [M_\nu, M_{\nu+1}]$ is

$$\ll M_\nu/(\varphi(a)\log^2 M_\nu),$$

assuming that $\eta < \delta/2$, say, and since clearly $\sum 1/\varphi(a) \ll \eta\log M_\nu$, we obtain, after some calculations,

$$\sum_{p\in\mathscr{B}_{j+1}} U_1(p) \leqslant c_1\eta\log(1/\delta).$$

Hence we proved that

(6.8) $$s(\mathscr{B}_j) \leqslant As(\mathscr{B}_{j+1})+B,$$

where

(6.9) $$B = c_1\eta\log(1/\delta).$$

Inequality (6.8) implies that

$$s(\mathscr{B}_0) \leqslant A^t s(\mathscr{B}_t)+B(1+A+A^2+\ldots+A^{t-1}).$$

Therefore, using (6.5) and (6.4), we obtain

(6.10) $$(1/x)E_t(x|\mathscr{B}_t) \leqslant A^t s(\mathscr{B}_t)+B(1+A+\ldots+A^{t-1}),$$

and so by (6.2) we obtain

(6.11) $$(1/x)A_t(x|\mathscr{B}_t) \leqslant (1+A+\ldots+A^t)s(\mathscr{B}_t)+B\Big(\sum_{j=0}^{t-1}(t-j)A^j\Big).$$

2. We are now in a position to prove that (5.2) is satisfied. Hence let

$$\lim_{x\to\infty}(1/x)\,\#\{n \leqslant x\colon P(R_j(n)) < x^{\delta_j}\} = e_j(\delta_j) \quad (j = 0, 1, \ldots, t).$$

Assume that $e_j(\delta_j)\to 0$ as $\delta_j\to 0$ has been proved for $j \leqslant t-1$. We shall prove it is also true for $j = t$.

Let $\varepsilon > 0$ be an arbitrary small number. Let us choose δ_{t-1} so that $e_{t-1}(\delta_{t-1}) < \varepsilon/2$ is satisfied. Let $\mathscr{D}$ be the set of those integers $n \leqslant x$ for which $P(R_t(n)) < x^{\delta_t}$. Write $\mathscr{D} = \mathscr{D}_1\cup\mathscr{D}_2$, where $P(R_{t-1}(n)) < x^{\delta_{t-1}}$ in $\mathscr{D}_1$, and $P(R_{t-1}(n)) \geqslant x^{\delta_{t-1}}$ in $\mathscr{D}_2$. Because of the hypothesis of induction, we have

$$\operatorname{card}(\mathscr{D}_1) \leqslant xe_{t-1}(\delta_{t-1})+o(x) < \varepsilon x$$

if x is large enough. It remains to estimate card $(\mathscr{D}_2)$. If $n \in \mathscr{D}_2$, then there exists a prime $q > x^{\delta_{t-1}}$, $q|R_{t-1}(n)$, $P(q-1) < x^{\delta_t}$. We shall now make use of the result of part 1. We define $\mathscr{B}_{t-1}$ as the set of the primes $q \in [x^{\delta_{t-1}}, x]$ for which $P(q-1) < x^{\delta_t}$. Then

$$s(\mathscr{B}_{t-1}) = \sum_{x^{\delta_{t-1}} < q < x;\, P(q-1) < x^{\delta_t}} 1/q. \tag{6.12}$$

It is clear that

$$\text{card}(\mathscr{D}_2) \leqslant A_{t-1}(x|\mathscr{B}_{t-1}).$$

We now substitute δ_{t-1} in the place of δ in (6.6) and (6.9). Further, let η be so small that $B\left(\sum_{j=0}^{t-1}(t-j)A^j\right) < \varepsilon/2$. Hence it follows, using (6.11), that

$$\text{card}(\mathscr{D}_2)/x \leqslant A^{t-1}s(\mathscr{B}_{t-1})+\varepsilon/2.$$

Collecting our estimates, we obtain

$$e_t(\delta_t) \leqslant A^t s(\mathscr{B}_{t-1})+\varepsilon.$$

It then follows from Section 2.8 that

$$\limsup_{\delta_t \to 0} e_t(\delta_t) \leqslant \varepsilon.$$

Since ε can be taken arbitrarily small, our result follows.

3. It remains to prove that (5.4) is satisfied. For this we assume that (5.4) is true for $\mathscr{A}_0^*, \ldots, \mathscr{A}_{t-1}^*$. We show that this fact implies that it is true for $\mathscr{A}_t^*$. Let $0 < \varepsilon < \delta$ be fixed and let $\mathscr{S}$ be the set of those prime pairs (q_1, q_2) for which $q_1/q_2 < x^\varepsilon$, $x^\delta \leqslant q_2 < q_1$, $q_1 q_2 \leqslant x$. Let η be small (depending on ε), M_η be the set of those integers $n \leqslant x$ for which $p|R_t(n)$ and $p \in D_\eta \cup E_\eta$. Here D_η is the set of primes $p \in [x^\delta, x]$ such that $p-1$ does not have prime divisors in the interval $[x^\eta, x^{\delta/2}]$; E_η is the set of primes $p \in [x^\delta, x]$ such that $p-1$ contains two prime divisors satisfying $q_1 \geqslant q_2 > x^{\delta/2}$, $q_2 \geqslant q_1^{1-\eta}$. By using parts 1 and 2 above, one can show that

$$\lim_{\eta\to 0}\limsup_{x\to\infty}\frac{A_t(x|D_\eta)}{x} = 0, \quad \lim_{\eta\to 0}\limsup_{x\to\infty}\frac{A_t(x|E_\eta)}{x} = 0.$$

To see this, it is enough to observe that

$$\lim_{\eta\to 0}\limsup_{x\to\infty}\Big(\sum_{p\in D_\eta}\frac{1}{p}\Big) = 0, \quad \lim_{\eta\to 0}\limsup_{x\to\infty}\Big(\sum_{p\in E_\eta}\frac{1}{p}\Big) = 0,$$

estimates which follow easily from known sieve results. Thus we have

$$\text{card}(\mathscr{M}_j) \leqslant (\tau(\eta)+o_x(1))x$$

where $\tau(\eta)\to 0$ as $\eta \to 0$. Let $\mathscr{N}_\eta = \{1, 2, \ldots, [x]\}\backslash \mathscr{M}_\eta$. Let $H_j(x|\mathscr{S})$ be the number of integers $n \in \mathscr{N}_\eta$ for which there exists $(q_1, q_2) \in \mathscr{S}$ satisfying $q_1 q_2 | R_j(n)$, but for which there exists no pair $(q_3, q_4) \in \mathscr{S}$ such that $q_3 q_4 | R_{j-1}(n)$. Let

$$K_t(x|\mathscr{S}) = \sum_{j=0}^{t} H_j(x|\mathscr{S}).$$

We shall prove that

$$\lim_{\varepsilon\to 0}\limsup_{x\to\infty}\frac{K_t(x|\mathscr{S})}{x} = 0. \tag{6.13}$$

This will imply (5.4) and end the proof of the theorem.

We have

$$H_0(x|\mathscr{S}) \leqslant x \sum_{(q_1,q_2)\in\mathscr{S}} \frac{1}{q_1 q_2} \leqslant x \sum \frac{1}{q_1}\Big(\sum\frac{1}{q_2}\Big)$$
$$\leqslant x \log\frac{1}{1-\varepsilon/\delta}\log\frac{1}{\delta}+o(x),$$

which proves that

$$\lim_{\varepsilon\to 0}\limsup_{x\to\infty}\frac{H_0(x|\mathscr{S})}{x} = 0.$$

Let now $1 \leqslant j \leqslant t$ and consider $H_j(x|\mathscr{S})$. Let n be counted in $H_j(x|\mathscr{S})$. Then $q_1 q_2 | n$ for a suitable choice of $(q_1, q_2) \in \mathscr{S}$, but there exist no $(q_3, q_4) \in \mathscr{S}$ such that $q_3 q_4 | R_{j-1}(n)$. Let us fix $(q_1, q_2) \in \mathscr{S}$. Then the following cases can occur:

(a) $q_1 | R_{j-1}(n)$, $q_2 \nmid R_{j-1}(n)$,

(b) $q_2 | R_{j-1}(n)$, $q_1 \nmid R_{j-1}(n)$,

(c) $q_2 \nmid R_{j-1}(n)$, $q_1 \nmid R_{j-1}(n)$.

We split the sum $H_j(x|\mathscr{S}) = H_j^{(a)}(x|\mathscr{S})+H_j^{(b)}(x|\mathscr{S})+H_j^{(c)}(x|\mathscr{S})$, according to these three cases.

We first consider the case (c). Let $U_{j-1}(q_i)$ be the set of primes $\{r_1^{(i)}, r_2^{(i)}, \ldots\}$ for which $q_i | r_\nu^{(i)}-1$ for every positive integer ν and which come up as divisors of $R_{j-2}(n)$ for at least one $n \in \mathscr{N}_\eta$. Assume that $\varepsilon < \eta\delta$. Then $U_{j-1}(q_1) \cap U_{j-1}(q_2) = \emptyset$. Indeed, if r were an element of the intersection, then it would imply that $q_1 q_2 | r-1$. But then $r \in E_\eta$, which is impossible because $n \in \mathscr{N}_\eta$. So there exist $r^{(1)} \in U_{j-1}(q_1)$, $r^{(2)} \in U_{j-1}(q_2)$ such that $r^{(1)} r^{(2)} | R_{j-1}(n)$. Furthermore, it is clear that $r^{(l)} \nmid R_{j-2}(n)$ $(l = 1, 2)$. Starting now from the set $X_{j-1} = U_{j-1}(q_1)$, define X_{j-2}^* as the set of those primes $p \in [x^\delta, x]$ for which there exists $r \in X_{j-1}$, $r|p-1$, and which do not belong to D_η. Now let $Y_{j-1} = U_{j-1}(q_2)$ and Y_{j-2}^* be defined by Y_{j-1} exactly in the same way as X_{j-2}^* was defined from $U_{j-1}(q_1)$. Further, let $Z_{j-2} = X_{j-2}^* \cap Y_{j-2}^*$, $X_{j-2} = X_{j-2}^*\backslash Z_{j-2}$, $Y_{j-2} = Y_{j-2}^*\backslash Z_{j-2}$. Let $X_{j-1}, \ldots, X_0$, $Y_{j-3}, \ldots, Y_0$, $Z_{j-3}, \ldots, Z_0$ be defined by the same process.

If n is counted, then the following cases can occur:

(1) there exist $p \in X_0$, $q \in Y_0$ such that $pq|n$,

$(2)_s$ there exists s, $0 \leq s \leq j-3$, and $p \in Z_s$ such that $p|R_s(n)$.
Since $X_0 \cap Y_0 = \emptyset$, then we have at most

$$x\left(\sum_{p\in X_0}\frac{1}{p}\right)\left(\sum_{q\in Y_0}\frac{1}{q}\right)$$

integers $n \notin D_\eta$ for which (1) holds. But

$$\sum\frac{1}{p} \leq \sum_{q\in X_1}\ \sum_{p\equiv 1 \pmod q;\, p>q^{1+\eta};\, p\in X_0}\frac{1}{p} \leq B\sum_{q\in X_1}\frac{1}{q},$$

where B is defined in (6.9). Continuing this process, we obtain

$$\sum_{p\in X_0}\frac{1}{p} \leq B^{j-1}\sum_{r\in X_{j-1}}\frac{1}{r}.$$

Since

$$\sum_{r\in X_{j-1}}\frac{1}{r} \leq \sum_{r\equiv 1 (\mathrm{mod}\, q_1);\, r>q_1^{1+\eta}}\frac{1}{r} < \frac{B}{q_1},$$

we get

$$\sum_{p\in X_0}\frac{1}{p} < \frac{B^j}{q_1}.$$

Similarly, we obtain

$$\sum_{q\in Y_0}\frac{1}{q} < \frac{B^j}{q_2}.$$

Let us now consider the case $(2)_s$. If $p \in Z_s$, then there exist $t^{(1)} \in X_{s+1}$, $t^{(2)} \in Y_{s+1}$ such that $t^{(1)}t^{(2)}|p-1$. Since $t^{(1)}t^{(2)} > p^{1-\eta}$ is excluded, we must have $p \nmid D_\eta$, and therefore

$$\sum_{p\in Z_s}\frac{1}{p} \leq \sum_{t^{(1)},t^{(2)}}\ \sum_{p\equiv 1 (\mathrm{mod}\, t^{(1)}t^{(2)});\, p>(t^{(1)}t^{(2)})^{1+\eta}}\frac{1}{p} \leq B\left(\sum_{t^{(1)}\in X_{s+1}}\frac{1}{t^{(1)}}\right)\left(\sum_{t^{(2)}\in Y_{s+1}}\frac{1}{t^{(2)}}\right).$$

Proceeding as above, we get

$$\sum\frac{1}{t^{(1)}} < \frac{B^{t-(s+1)}}{q_1}, \quad \sum\frac{1}{t^{(2)}} < \frac{B^{t-(s+1)}}{q_2},$$

and so

$$\sum_{p\in Z_s}\frac{1}{p} < \frac{B^{2(t-s)-1}}{q_1 q_2}.$$

Collecting our estimates and summing over $(q_1, q_2) \in \mathscr{S}$, we finally obtain

$$\lim_{\varepsilon\to 0}\limsup_{x\to\infty}\frac{1}{x}H_j^{(c)}(x|\mathscr{S}) = 0.$$

Since there is essentially no difference between the treatment of the cases (a) and (b), we will only consider case (b). So let q_1 and q_2 be fixed. Let s be the smallest integer such that $q_2|R_s(n)$, $0 \leq s \leq j-1$. Let W_{j-1} be the set of those primes $r \nmid D_\eta$ such that $q_1|r-1$, $r \leq x$. Let W_{j-2} be the set of those primes $u \leq x$ such that $r|u-1$ for some $r \in W_{j-2}$, $u \nmid D_\eta$. Continuing this process, we define $W_{j-3}, \ldots, W_s$. If n is counted, then there exists $r^{(s)} \in W_s$ such that $r^{(s)}|R_s(n)$, $q_2|R_s(n)$ and such that $r^{(s-1)} \nmid R_{s-1}(n)$, $q_2 \nmid R_{s-1}(n)$. It follows that the estimation

$$\#\{n \in \mathscr{N}_\eta:\ r^{(s)}q_2|R_s(n),\ \text{case (c)}\} \leq B^{2s}/(q_2 r^{(s)})$$

can be deduced as earlier. Moreover, since

$$\sum_{r^{(s)}\in W_s} 1/r^{(s)} < B^{t-s}/q_1,$$

we have

$$\lim_{\varepsilon\to 0}\limsup_{x\to\infty}\frac{1}{x}H_j^{(b)}(x|\mathscr{S}) = 0.$$

Similarly we can deduce that

$$\lim_{\varepsilon\to 0}\limsup_{x\to\infty}\frac{1}{x}H_j^{(a)}(x|\mathscr{S}) = 0.$$

Summing up for $j \leq t$, we see that for $d(\varepsilon, \delta)$ defined in (5.3),

$$\limsup_{\varepsilon\to 0} d(\varepsilon, \delta) \leq \tau(\eta).$$

But since this is true for every $\eta > 0$ and $\tau(\eta) \to 0$ as $\eta \to 0$, it follows that (5.4) is true.

This ends the proof of Theorem 3.

7. Further applications. There are other cases in which we can apply Theorem 2 successfully. The following results, which we state as theorems, can be obtained by using the theorem mentioned in 2.5 and from other known sieve results.

THEOREM 4. *Let $\mathscr{A}_l = \{p+l:\ p$ is prime$\}$ be the set of shifted primes. Then, if $l \neq 0$, the conditions of Theorem 2 hold. Consequently, if $(\mathscr{H}_\Delta)$ is true for $0 < \Delta < 1$, then $u(p+l) \to 0$ as $p \to \infty$ for almost all primes p.*

THEOREM 5. *Let $k \geq 3$, $(0 <)\, l_1 < l_2 < \ldots < l_s\, (< k)$ be integers, coprime to k, $(l_j, k) = 1$ $(j = 1, \ldots, s)$. Let $\mathscr{B}$ be the set of those integers n for which all the prime divisors belong to the union of the sets $\{p:\ p \equiv l_j \pmod k\}$. Let $\mathscr{B}_l = \mathscr{B} \dotplus l = \{n+l:\ n \in \mathscr{B}\}$. Then the conditions of Theorem 2 are satisfied for $\mathscr{A} = \mathscr{B}_l$ $(l = 0, \pm 1, \pm 2, \ldots)$.*

Theorem 6. *Let $K(x)$ be a polynomial with rational roots, taking on positive integer values for every $n > n_0$. Let $\mathscr{A} = \{K(n): n > n_0\}$. Then the conditions of Theorem 2 hold.*

Remark 7. We are unable to prove (5.4) if $\mathscr{A} = \{n^2+1: n = 1, 2, \ldots\}$.

8. The distribution of the large values of $f(n)/L(P(n))$. Let

$$v(n) \overset{\text{def}}{=} u(n)+1 = \frac{f(n)}{L(P(n))} = \sum_{q|n} \frac{L(q)}{L(P(n))}, \tag{8.1}$$

$$Q(K) \overset{\text{def}}{=} \limsup_{x\to\infty} \frac{1}{x} \# \{n \leqslant x:\ v(n) \geqslant K\}. \tag{8.2}$$

In this section we shall give necessary and sufficient conditions for L which guarantee the fulfilment of

$$\lim_{K\to\infty} Q(K) = 0. \tag{8.3}$$

We first define a function which will play an important role, namely

$$a(x) \overset{\text{def}}{=} \sum_{q\leqslant x} L(q)/q. \tag{8.4}$$

Using the Turán–Kubilius inequality (stated in (2.7)), we obtain

$$\sum_{n\leqslant x} \left(\frac{f(n)}{L(x)} - \frac{a(x)}{L(x)}\right)^2 < cx \sum_{q\leqslant x} \frac{1}{q} \left(\frac{L(q)}{L(x)}\right)^2. \tag{8.5}$$

Since $L(q)/L(x) \leqslant 1$, the right-hand side of (8.5) is less than $cx\,a(x)/L(x)$, hence we have

$$\# \left\{n \leqslant x:\ \frac{f(n)}{L(x)} < \frac{1}{2} \frac{a(x)}{L(x)}\right\} < 4cx \frac{L(x)}{a(x)}. \tag{8.6}$$

Since $f(n)/L(x) \leqslant v(n)$, we obtain

$$\# \left\{n \leqslant x:\ v(n) < \frac{1}{2} \frac{a(n)}{L(x)}\right\} < \frac{4cx}{a(x)/L(x)}. \tag{8.7}$$

If $a(x)/L(x)$ is not bounded and $x_\nu \to \infty$ is a sequence of real numbers such that

$$z_\nu \overset{\text{def}}{=} a(x_\nu)/L(x_\nu) \to \infty, \qquad \text{as } \nu\to\infty,$$

then, from (8.7), we obtain

$$\# \{n \leqslant x_\nu:\ v(n) \geqslant z_\nu/2\} > (1-\varepsilon)\, x_\nu,$$

for every $\varepsilon > 0$, provided ν is large enough.

Thus the condition

$$a(x) \leqslant cL(x) \tag{8.8}$$

is necessary to guarantee (8.3). Now assume that it holds and let $0 < \eta < 1$ be a constant. Then

$$a(x) \geqslant \sum_{x^\eta < q \leqslant x} \frac{L(q)}{q} \geqslant L(x^\eta) \left(\log \frac{1}{\eta} + O\left(\frac{1}{\eta \log x}\right)\right). \tag{8.9}$$

From (8.8) and (8.9) we infer that

$$L(x^\eta)/L(x) \leqslant 1/2 \quad \text{if } \eta \text{ is small and } x > x_0. \tag{8.10}$$

Let η be fixed. Using (8.10) and repeating the arguments used in the proofs of Lemmas 2 and 3, it follows that there exists a suitable positive $\varkappa$, depending only on η, such that

$$\frac{L(y)}{L(x)} \leqslant \left(\frac{\log y}{\log x}\right)^\varkappa \quad \text{whenever } x_0 \leqslant y \leqslant x^\eta. \tag{8.11}$$

Finally, note that it is easy to prove that (8.8) is equivalent to

$$\int_e^x \frac{L(u)}{u \log u}\, du \leqslant c_1 L(x), \tag{8.12}$$

where c_1 is a suitable positive constant.

Theorem 7. *Relation* (8.3) *holds if and only if* (8.12) *is satisfied. Moreover, if* (8.12) *is satisfied, then, for every $a > 0$,*

$$\sum_{n\leqslant x} e^{av(n)} = O(x). \tag{8.13}$$

Consequently, there exists a positive constant $C = C(a)$ such that

$$Q(K) \leqslant C_a e^{-Ka}.$$

Proof. We have already shown above that "(8.3) $\Rightarrow$ (8.12)". So assume that (8.12) holds, that is, that (8.8) is true. To prove (8.13), we may ignore the integers $n \leqslant x$ such that $P(n) \leqslant \exp(\sqrt{\log x})$, since their contribution to (8.13) is $o(x)$. Let x_0, η, $\varkappa$ be determined from (8.11). Further, let

$$v(n) = v_1(n)+v_2(n)+v_3(n),$$

where in their definitions (see (8.1)), we sum over $q \leqslant x_0$, $x_0 < q \leqslant P(n)^\eta$, $P(n)^\eta < q < P(n)$, respectively. It is clear that $v_1(n)$ tends to 0 uniformly. Therefore, using the Cauchy–Schwarz inequality, it is enough to prove that both the estimates

$$\sum_{n\leqslant x} e^{2av_j(n)} = O(x) \quad (j = 2, 3) \tag{8.14}$$

hold.

For $j = 2$, we split the sum (8.12) into subsums according to the largest prime factor belonging to

$$I_k \stackrel{\text{def}}{=} [x^{1/2^k}, x^{1/2^{k-1}}], \quad k = 1, 2, \ldots$$

and so

$$\Sigma_k \stackrel{\text{def}}{=} \sum_{n \leqslant x} e^{4a l_k(n)} \leqslant x \prod_{q \leqslant x^{1/2^{k-1}}} \left(1 + \frac{e^{4a2^{k\varkappa}(\log q/\log x)^\varkappa} - 1}{q}\right).$$

Using (2.4), we observe that the product on the right-hand side is bounded by a constant that depends on $\varkappa$ but not on k.

Furthermore, by the Cauchy–Schwarz inequality combined with the inequality

$$\Psi(x, x^{1/2^{k-1}}) \leqslant x \exp(-c2^{k-1}),$$

we deduce that

(8.15) $$\sum_{n \leqslant x} e^{2av_2(n)} \leqslant \sum_k \sum_{P(n) \in I_k} e^{2av_2(n)} \leqslant \sum_k \Psi(x, x^{1/2^{k-1}})^{1/2} \Sigma_k^{1/2} \ll x \sum_k \exp(-c2^{k-2}) \ll x.$$

It remains to consider the case $j = 3$.

If n contains exactly r distinct prime factors $q_1, \ldots, q_r$ in the interval $p^\eta \leqslant q_j < p$, then $v_3(n) \leqslant r$. Therefore the left-hand side of (8.14), in the case $j = 3$, is bounded by

(8.16) $$\sum_{r=0}^{\infty} e^{2ar} R_r(x),$$

where

(8.17) $$R_r(x) \leqslant \sum \Psi\left(\frac{x}{pq_1 \ldots q_r}, p^\eta\right).$$

Now $pq_1 \ldots q_r > \sqrt{x}$ implies that $p > x^{1/r(r+1)}$; furthermore $p^{1+r\eta} \leqslant pq_1 \ldots q_r < x$, and so $p < x^{1/(1+r\eta)}$. Therefore the contribution of the terms satisfying $\sqrt{x} < pq_1 \ldots q_r$ is less than

$$\frac{x}{r!} \sum_p \frac{1}{p} \left(\sum_{p^\eta < q < p} \frac{1}{q}\right)^r < \frac{(2\log(1/\eta))^r}{r!} \left(\sum \frac{1}{p}\right) x \leqslant 2\left(\log \frac{2}{\eta}\right) \frac{(2\log(1/\eta))^r}{r!} x.$$

For the other terms we have

$$\Psi\left(\frac{x}{pq_1 \ldots q_r}, p^\eta\right) \leqslant \frac{x}{pq_1 \ldots q_r} \exp\left(-\frac{c \log x}{2 \log p}\right).$$

Summing first for $q_1, \ldots, q_r$, and after for p, we get

$$R_r(x) \leqslant (A+B) \frac{(2\log(1/r))^r}{r!} x,$$

where $A = 2\log(2/\eta)$, and B is a number such that

$$\sum_{p \leqslant x} \frac{1}{p} \exp\left(-\frac{c \log x}{2 \log p}\right) \leqslant B.$$

Substituting this inequality in (8.16), we obtain (8.14) in the case $j = 2$. This ends the proof.

Remark 8. It is interesting to note that Theorem 7 remains true if we replace the whole set of integers by the set of values of an arbitrary polynomial or by the set of shifted primes.

9. On the limit distribution of $v(n)$. Let

(9.1) $$F_x(y) \stackrel{\text{def}}{=} \frac{1}{x} \# \{n \leqslant x \colon v(n) < y\},$$

where $v(n)$ is defined in (8.1). We say that v has a *limit distribution* if

(9.2) $$\lim_{x \to \infty} F_x(y) = F(y)$$

exists for almost all y, and F is a distribution function.

It would be nice if one could characterize those functions L for which v has a limit distribution. We are unable to solve this problem in general.

We shall nevertheless discuss what happens when v has a limit distribution. First of all, in that case, (8.3) is true, and hence the conditions of Theorem 7 are satisfied. Thus (8.13) is true, and so by (9.2) we have $1 - F(y) = O(e^{-ay})$ as $y \to \infty$. From this we can deduce that

(9.3) $$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leqslant x} v(n)^k = c_k$$

exists for every $k = 1, 2, \ldots$ Indeed, if $A > 0$, we have

$$\frac{1}{x} \sum_{n \leqslant x, v(n) \leqslant A} v(n)^k = \int_1^A y^k \, dF_x(y) = \int_1^A y^k \, dF(y) + o_x(1).$$

Furthermore,

$$\frac{1}{x} \sum_{n \leqslant x, v(n) > A} v(n)^k \leqslant \frac{1}{xA^k} \sum_{n \leqslant x} v(n)^{2k} \leqslant \frac{d_{2k}}{A^k},$$

for a suitable constant d_{2k}, and $\int_A^\infty y^k \, dF(y) \to 0$ as $A \to \infty$. Combining these, (9.3) follows immediately. Taking into account the fact that

$$\sum_{j=0}^{k} (av(n))^j / j! \leqslant e^{av(n)},$$

we find, using Theorem 7, that $\sum_{j=0}^{\infty} c_j a^j/j!$ is convergent. But, as is well known, in this case the existence of all moments given in (9.3) is sufficient for the existence of the limit distribution.

On the other hand, it is clear that

(9.4) $$\sum_{n\leqslant x} v(n) = [x] + \sum_{q<p\leqslant x} \frac{L(q)}{L(p)} \Psi\left(\frac{x}{pq}, p\right).$$

Taking into account (2.2), for every fixed $0 < \lambda < 1$, we have

$$\Psi\left(\frac{x}{pq}, p\right) = (1+o(1))\frac{x}{pq}\varrho\left(\frac{\log x - \log p - \log q}{\log p}\right)$$

uniformly for $p > (x/(pq))^{\lambda}$. Furthermore, the contribution of the terms $q < p < x^{\lambda_1}$ in the sum on the right-hand side of (9.4) is $\leqslant Q(\lambda_1)x$, where $Q(\lambda_1) \to 0$ as $\lambda_1 \to 0$. Hence, from the existence of (9.3) for $k = 1$, we can deduce that

(9.5) $$\sum_{q<p<x} \frac{L(q)}{L(p)\,qp}\varrho\left(\frac{\log x - \log p - \log q}{\log p}\right) \to c_1 - 1 \quad \text{as } x \to \infty.$$

Clearly (9.5) is equivalent to

(9.6) $$\int_{1\leqslant \eta<\xi<\log x} \frac{L(e^{\eta})}{L(e^{\xi})}\varrho\left(\frac{\log x - \xi - \eta}{\xi}\right)\frac{d\eta\, d\xi}{\eta\ \ \xi} \to c_1 - 1 \quad \text{as } x \to \infty.$$

It is clear that (9.6) implies (9.3) in the case $k = 1$.

Note that the relations (9.3) for an arbitrary integer $k > 1$ can be expressed by relations equivalent to (9.6), but these become very complicated for large values of k, and it becomes very difficult to characterize the functions L which satisfy them.

In the next section, we consider a somewhat easier problem.

10. On the distribution of $f(n)/L(x)$. Let

(10.1) $$v_x(n) \overset{\text{def}}{=} f(n)/L(x)$$

and

$$F_x(y) \overset{\text{def}}{=} \frac{1}{x}\,\#\{n \leqslant x:\ v_x(n) < y\}.$$

We say that $v_x(n)$ has a *limit distribution* F if

(10.2) $$F(y) = \lim_{x\to\infty} F_x(y)$$

exists for almost all y.

Using the standard theory of limit distributions for additive functions, as presented in Chapters 16–18 of Elliott [4], one can obtain necessary and sufficient conditions for the existence of the limit distribution $F(y)$. More precisely, one can prove that the existence of the limit

(10.3) $$\lim_{x\to\infty} \frac{a(x)}{L(x)} = A,$$

(where $a(x)$ is defined in (8.4)) with

(10.4) $$A = \lim_{x\to\infty} \frac{1}{x}\sum_{n\leqslant x} v_x(n) \geqslant 0$$

is a necessary condition for the fulfilment of (10.2).

On the other hand, if (10.3) holds with $A = 0$, then $F(y)$ exists and has a maximal jump at $y = 0$, that is,

$$F(y) = \begin{cases} 0 & \text{if } y \leqslant 0, \\ 1 & \text{if } y > 0. \end{cases}$$

Finally, if (10.3) holds with $A > 0$, then it is easy to see that

(10.5) $$a(x) = (1+o(1))\int_2^x \frac{L(u)}{u\log u}\,du$$

and thus, using Lemma 1, that

(10.6) $$L(x) = (\log x)^{1/A} H(x),$$

where H is a very slowly oscillating function.

Therefore, with $L(x)$ as in (10.6) (set $\alpha = 1/A > 0$) and introducing the following functions:

$$k(n) \overset{\text{def}}{=} \sum_{q|n} L(q), \qquad K_x(n) \overset{\text{def}}{=} \frac{k(n)}{L(x)},$$

$$V_x(n) \overset{\text{def}}{=} \sum_{q|n}\left(\frac{\log q}{\log x}\right)^{\alpha}, \qquad t(n) \overset{\text{def}}{=} \frac{k(n)}{L(P(n))},$$

$$T(n) \overset{\text{def}}{=} \frac{1}{(\log P(n))^{\alpha}}\sum_{q|n}(\log q)^{\alpha} \qquad (n \geqslant 2),$$

one can prove, using Theorems 18.1 and 18.2 (along with Lemma 1.4) of Elliott [4], the following results:

THEOREM 8. *Assuming the conditions stated above, $K_x(n)$ has a limit distribution F, that is,*

$$F(y) = \lim_{x\to\infty}\frac{1}{x}\,\#\{n \leqslant x:\ K_x(n) < y\}$$

for almost all y. Furthermore, F does not depend on H, so $K_x(n)$ is distributed as $V_x(n)$.

THEOREM 9. *Under the conditions stated in Theorem 8,*

$$\lim_{x\to\infty} \frac{1}{x} \# \{n \leqslant x:\ t(n) < y\} = G(y)$$

exists for almost all y; furthermore, G is a distribution function and it does not depend on H. The same result holds when replacing $t(n)$ by $T(n)$.

Acknowledgement. The authors wish to thank the referee for some helpful comments.

References

[1] N. G. de Bruijn, *On the number of positive integers $\leqslant x$ and free of prime factors $v > y$*, Koninkl. Nederl. Akademie Van Wetenschappen, Series A, 54 (1951), 49–60.
[2] J. M. De Koninck, I. Kátai and A. Mercier, *Additive functions and the largest prime factors of integers*, J. Number Theory 33 (1989), 293–310.
[3] J. M. De Koninck et A. Mercier, *Les fonctions arithmétiques et le plus grand facteur premier*, Acta Arith. 52 (1989), 27–48.
[4] P. D. T. A. Elliott, *Probabilistic Number Theory*, I and II, Springer-Verlag, 1979.
[5] H. Halberstam and H. E. Richert, *Sieve Methods*, L.M.S. Monograph, Academic Press, 1975.

Jean-Marie De Koninck
DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE
UNIVERSITÉ LAVAL
Québec, G1K 7P4
Canada

Imre Kátai
EOTVOS LORÁND UNIVERSITY
COMPUTER CENTER
1117 Budapest, Bogdánfy u. 10/B
Hungary

Armel Mercier
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DU QUÉBEC
Chicoutimi, G7H 2B1
Canada

Received on 18.4.1989
and in revised form on 8.8.1989 (1927)