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RANDOM SUMS RELATED TO PRIME DIVISORS OF AN INTEGER

J. M. De Koninck and **A. Ivić**

To the memory of J. Karamata

Abstract. The asymptotic behaviour of the sums $\sum_{2\le n\le x}(1/\omega(n))\sum_{p|n} f(p)$ is investigated. Here $\omega(n)$ denotes the number of distinct prime factors of an integer n, p denotes primes, and $f(x) = x^\rho L(x)$, where $L(x)$ is a slowly oscillating function. The above sums naturally arise when one defines random sums related to the prime divisors of an integer from the probabilistic viewpoint. Three different types of asymptotic results, corresponding to the cases $\rho < 0$, $\rho > 0$ and $\rho = 0$, are derived.

1. Introduction

Let $\omega(n)$ denote the number of distinct prime divisors of a positive integer n and let $p_1(n) < p_2(n) < \dots < p_\omega(n)$ be the distinct prime divisors of n. Let $f(x) = x^\rho L(x)$ be a regularly varying function; here ρ is a real number and $L(x)$ is a slowly varying function in the sense of Karamata (that is, a continuous function defined for $x \ge 2$ and satisfying $\lim_{x\to\infty} \dfrac{L(ax)}{L(x)} = 1$, for each fixed $a > 0$). It is known (see Seneta [6]) that, for large x, such functions can be written in the form

$$L(x) = A(x) \exp\left(\int_{x_0}^{x} \frac{\eta(t)}{t}\,dt\right), \tag{1.1}$$

where $A(x)$ tends to a positive constant $A > 0$, $\eta(t)$ is continuous and $\lim_{t\to\infty}\eta(t) = 0$. For each positive integer $n \ge 2$, we define the random value $f_r(n) = f(p_j(n))$, where $p_j(n)$ $(j = 1, 2, \dots, \omega(n))$ is any of the prime divisors of n, picked with equal probabilities, and set

$$S_r(x) = \sum_{2\le n\le x} f_r(n). \tag{1.2}$$

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Integration by parts then yields

$$\begin{aligned} I &\stackrel{\text{def}}{=} \int_{3/2}^{x} \frac{t^{\rho}L(t)}{\log t}\,dt \\ &= \frac{x^{\rho+1}L(x)}{(\rho+1)\log x} + O(1) - \int_{3/2}^{x} \frac{t^{\rho+1}}{\rho+1}\left(\frac{L(t)\eta(t)}{t\log t} - \frac{L(t)}{t\log^2 t}\right)dt \\ &= \frac{x^{\rho+1}L(x)}{(\rho+1)\log x} + O\left(\frac{x^{\rho+1}L(x)}{\log^2 x}\right) + O\left(\widehat{\eta}(x)I\right), \end{aligned}$$

and $\lim_{x\to\infty}\widehat{\eta}(x) = 0$, since $\lim_{x\to\infty}\eta(x) = 0$ by hypothesis. Hence

$$I = \frac{x^{\rho+1}L(x)}{(\rho+1)\log x}\left(1 + O\left(\widehat{\eta}(x)\right) + O\left(\frac{1}{\log x}\right)\right),$$

and the lemma follows. We remark that for $\widehat{\eta}(x)$, we could have taken $\sup_{x^c\le t\le x}|\eta(t)|$ for any fixed $0 < c < 1$.

LEMMA 2. *If $L \in \mathcal{L}$, then there exists a real-valued function φ such that $\lim_{x\to\infty}\varphi(x) = +\infty$ and*

$$L\left(\frac{x}{t}\right) = L(x)\left\{1 + O\left(\sqrt{\widehat{\eta}(x)}\right)\right\}$$

uniformly for $1 \le t \le \varphi(x)$, where $\widehat{\eta}(x) = \sup_{\sqrt{x}\le t\le x}|\eta(t)|$. In particular, one can take

$$\varphi(x) = \min\left\{\sqrt{x}, \exp\left(1/\sqrt{\widehat{\eta}(x)}\right)\right\}.$$

Proof. This is a modified version of Lemma 1 of De Koninck - Mercier [4], which holds if $\eta(x)$ is assumed to be monotonic. Note that, with φ as above,

$$I \stackrel{\text{def}}{=} \left|\int_{x/t}^{x}\eta(u)\frac{du}{u}\right| \le \int_{x/\varphi(x)}^{x}|\eta(u)|\frac{du}{u} \le \widehat{\eta}(x)\int_{x/\varphi(x)}^{x}\frac{du}{u} = \widehat{\eta}(x)\log\varphi(x) \le \widehat{\eta}^{\frac{1}{2}}(x).$$

Since $\widehat{\eta}(x) \to 0$ as $x \to \infty$, we have that $0 \le I \le 1$ for $x \ge x_1$. Then using $e^x = 1 + O(|x|)$ for $|x| \le 1$, we obtain

$$\frac{L\left(\frac{x}{t}\right)}{L(x)} = \exp\left(-\int_{x/t}^{x}\eta(u)\frac{du}{u}\right) = 1 + O(I) = 1 + O\left(\widehat{\eta}^{\frac{1}{2}}(x)\right)$$

with the O-constant uniform in t. In particular, for $1 \le n \le \varphi(x)$, we have, uniformly in n,

$$L\left(\frac{x}{n}\right) = (1 + o(1))L(x) \qquad (x \to \infty).$$

LEMMA 3. *Let $L \in \mathcal{L}$ and $\delta > -1$. Then, for any fixed $D > x_0$ and $\varepsilon > 0$,*

$$\int_D^x u^{\delta}L(u)\,du = \left(\frac{1}{\delta+1} + O(\widehat{\eta}(x))\right)x^{\delta+1}L(x) + O\left(x^{\frac{\delta+1}{2}+\varepsilon}\right),$$

where $\widehat{\eta}(x) = \sup_{\sqrt{x}\le t\le x}|\eta(t)|$.

Proof. The proof is similar to that of Lemma 1. Integration by parts gives

$$\begin{aligned} I &\stackrel{\text{def}}{=} \int_D^x u^{\delta}L(u)\,du = \frac{u^{\delta+1}L(u)}{\delta+1}\Big|_D^x - \frac{1}{\delta+1}\int_D^x u^{\delta+1}L'(u)\,du \\ &= \frac{x^{\delta+1}L(x)}{\delta+1} + O(1) + O\left(\int_D^{\sqrt{x}} u^{\delta}L(u)du\right) + O\left(\int_{\sqrt{x}}^{x} u^{\delta}L(u)|\eta(u)|du\right) \\ &= \frac{x^{\delta+1}L(x)}{\delta+1} + O\left(\widehat{\eta}(x)I\right) + O\left(x^{\frac{\delta+1}{2}+\varepsilon}\right), \end{aligned}$$

since $L(u) \ll u^{\varepsilon}$, and the lemma follows. Note that as a corollary we obtain

$$\int_D^x u^{\delta}L(u)du = \left(\frac{1}{\delta+1} + o(1)\right)x^{\delta+1}L(x) \qquad (x \to \infty),$$

which is in fact equivalent to the statement that $L(x)$ is slowly varying by a classical theorem of Karamata (Th. 2.1 of [6]). The last relation shows, after an integration by parts, that as $x \to \infty$

$$\int_{x_0}^{x} t^{\alpha}\,dL(t) = o(x^{\alpha}L(x)) \qquad (\alpha > 0).$$

Thus we may obtain by the method of proof of Lemma 1 that, if $\rho > -1$ and $x \to \infty$,

$$\sum_{p\le x} p^{\rho}L(p) = \left(\frac{1}{\rho+1} + o(1)\right)\frac{L(x)x^{\rho+1}}{\log x}.$$

This is weaker than the statement of Lemma 1, but holds for most general slowly varying functions $L(x)$, whereas for Lemma 1 we assumed that $A(x)$ in (1.1) was a constant.

LEMMA 4. *Let $L \in \mathcal{L}$. Then*

(i) $\displaystyle\lim_{x\to\infty}\frac{1}{L(x)}\int_{x_0}^{x}\frac{L(u)}{u}\,du = +\infty,$

(ii) $I_L \in \mathcal{L}$.

Proof. We start with part (i). By Lemma 2, we have $L(u) \sim L(x)$ for $x/\varphi(x) \le u \le x$ and $\lim_{x\to\infty}\varphi(x) = +\infty$. Thus

$$\int_{x_0}^{x}\frac{L(u)}{u}\,du \ge \int_{x/\varphi(x)}^{x} L(u)\frac{du}{u} \sim L(x)\int_{x/\varphi(x)}^{x}\frac{du}{u} = L(x)\log\varphi(x),$$

which implies (i).

In order to prove (ii), we only need to prove that, as $x \to \infty$,

$$\eta_{I_L}(x) = x\frac{I_L'(x)}{I_L(x)} = o(1).$$

$$+(J+1)!\sum_{m\le yp^{-J-1}}\frac{1}{\omega(pm)(\omega(pm)+1)\dots(\omega(pm)+J+1)}.$$

In the last sum, we use the fact that, uniformly in p, $\omega(pm)=\omega(m)+O(1)$. For $J\ge M$, this sum then clearly contributes to the error term in (3.1). For the remaining sums in (3.2), we have, for any fixed $j, R\ge 0$,

(3.3)
$$\sum_{m\le x}\frac{1}{(\omega(m)+1)\dots(\omega(m)+j)}=\sum_{r=0}^{R}\frac{d_{r,j}x}{(\log\log x)^{j+r}}+O\left(\frac{x}{(\log\log x)^{R+j+1}}\right),$$

where the constants $d_{r,j}$ are effectively computable and $d_{0,j}=1$ for each j. The proof of (3.3) follows from the general methods of De Koninck-Ivić [2]. Namely their equation (2.22) gives an asymptotic formula for $\sum_{2\le n\le x}t^{\omega(n)}$ with the error term uniform in t for $|t|\le 1$. This is integrated $j-1$ times over t from $\varepsilon(x)\doteq(\log x)^{-c}$ (with suitable $c=c(j)>0$) to t, and then finally from $\varepsilon(x)$ to 1, producing (3.3). Using (3.2) and (3.3) in the expression for $\sum_1$, we see that we shall obtain sums of the type

$$x\sum_{p\le\exp(\sqrt{\log x})}p^{\rho-1}L(p)(\log\log(xp^{-j})^{-k})\qquad(j=0,1,\dots;\ k=1,2,\dots)$$

in which we replace $(\log\log(xp^{-j}))^{-k}$ by

$$(\log\log x)^{-k}\left\{1+\sum_{r=1}^{R}\binom{-k}{n}\left(\frac{\log\left(1-\frac{p^j}{\log x}\right)}{\log\log x}\right)^{r}+O\left(\left(\frac{\log p^j}{\log x\log\log x}\right)^{R+1}\right)\right\}$$

and simplify. Since $\rho<0$, all the series

$$\sum_p p^{\rho-1}L(p)(\log p^j)^r$$

are convergent for any fixed r, j, and the portion of the series for $p>\exp(\sqrt{\log x})$ will make a negligible contribution. This ends the proof of Theorem 1.

Remark 2. Note that one could obtain a further sharpening of (3.1). It could be done by using an asymptotic formula sharper than (3.3), essentially in the same way that Theorem 2.5 of [2] was sharpened to Corollary 5.3. The truncation of $\sum_1$ at $\exp(\sqrt{\log x})$ will work in that case too; for (3.1), we could have made the truncation at $\log^A x$ for some large $A>0$. Also note that Theorem 1 holds if $L(x)$ is of the most general form (1.1).

4. The case "$\rho>0$"

We now have to deal with "large" additive functions. In the case $f(x)=x$, this problem has been tackled by De Koninck - Ivić [3], who proved a sharp asymptotic formula for the summatory function of

$$P_*(n)=\frac{1}{\omega(n)}\sum_{p|n}p$$

(the function $P_*(n)$ may be called the average prime divisor of n) and some related arithmetical functions, such as

$$P^*(n)=\frac{1}{\Omega(n)}\sum_{p^\alpha\|n}\alpha p,$$

where $\Omega(n)$ stands for the total number of prime divisors of n.

THEOREM 2. *If $f(x)=x^\rho L(x)$, with $\rho>0$ and $L\in\mathcal{L}$, then as $x\to\infty$,*

(4.1)
$$\sum_{2\le n\le x}\frac{1}{\omega(n)}\sum_{p|n}f(p)=(C_\rho+o(1))\frac{x^{1+\rho}L(x)}{\log x},$$

where $C_\rho=\frac{1}{1+\rho}\sum_{n=1}^{\infty}\frac{1}{(\omega(n)+1)n^{1+\rho}}$.

Proof. Clearly we have $\omega(pm)=\omega(m)$ if $p|m$. Therefore

(4.2)
$$\begin{aligned}S(x)&=\sum_{2\le n\le x}\frac{1}{\omega(n)}\sum_{p|n}f(p)=\sum_{p\le x}f(p)\sum_{m\le x/p}\frac{1}{\omega(pm)}\\&=\sum_{m\le x}\left\{\frac{1}{\omega(m)+1}\sum_{p\le x/m,\,(p,m)=1}f(p)+\frac{1}{\omega(m)}\sum_{p\le x/m,\,p|m}f(p)\right\}\\&=\sum_{m\le x}\left\{\frac{1}{\omega(m)+1}\sum_{p\le x/m}f(p)-\frac{1}{\omega(m)+1}\sum_{p\le x/m,\,p|m}f(p)\right.\\&\qquad\left.+\frac{1}{\omega(m)}\sum_{p\le x/m,\,p|m}f(p)\right\}\\&=\sum_{m\le x}\frac{1}{\omega(m)+1}\sum_{p\le x/m}f(p)+\sum_{m\le x}\frac{1}{\omega(m)(\omega(m)+1)}\sum_{p\le x/m,\,p|m}f(p)\\&=\sum\nolimits_1+\sum\nolimits_2,\end{aligned}$$

say. Now $\sum_2$ is of smaller order than the main term on the right hand side of (4.1). To see this, we write

(4.3)
$$\sum\nolimits_2\ll\sum_{m\le x}\sum_{p\le x/m,\,p|m}f(p)=\sum_{m\le\sqrt{x}}\sum_{p\le x/m,\,p|m}f(p)+\sum_{\sqrt{x}<m\le x}\sum_{p\le x/m,\,p|m}f(p).$$

Since $\rho>0$, $f(x)$ is increasing for $x\ge x_0$. Thus using $L(x)\ll_\varepsilon x^\varepsilon$ we have

$$\sum\nolimits_2\ll\log x\sum_{m\le\sqrt{x}}f(m)+\log x\sum_{\sqrt{x}<m\le x}f(\sqrt{x})\ll x^{1+\frac{\rho}{2}}L(\sqrt{x})\log x=o(x^{1+\frac{3\rho}{4}}),$$

as $x\to\infty$. This proves our claim about the size of $\sum_2$.

$$= \sum_{p\le x/\xi(x)} L(p) \sum_{m\le x/p} \frac{1}{\omega(pm)} + \sum_{x/\xi(x)<p\le x} L(p) \sum_{m\le x/p} \frac{1}{\omega(pm)}$$
$$= \sum\nolimits_1 + \sum\nolimits_2,$$

say. Here $\xi(x)$ is a function, to be suitably chosen later, which satisfies

(5.1) $$\lim_{x\to\infty} \xi(x) = +\infty, \qquad \xi(x) \le \sqrt{x} \quad (x \ge x_1).$$

Suppose now either (i) or (ii) holds. We use the elementary estimate

$$\sum_{p\le x} \frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log^A x}\right),$$

which holds for any fixed $A > 0$. With $A = 2$ we obtain, since $L(x)$ is increasing and $\xi(x) \le \sqrt{x}$,

$$\sum\nolimits_2 = \sum_{x/(\xi(x))<p\le x} L(p) \sum_{m\le x/p} \frac{1}{\omega(mp)} \le x \sum_{x/(\xi(x))<p\le x} \frac{L(p)}{p}$$
$$\le xL(x) \sum_{x/(\xi(x))<p\le x} \frac{1}{p} = xL(x)\left\{\log \frac{1}{1-\frac{\log\xi(x)}{\log x}} + O\left(\frac{1}{\log^2 x}\right)\right\}$$
$$\ll \frac{xL(x)\log\xi(x)}{\log x} + \frac{xL(x)}{\log^2 x}.$$

But

$$I_L(x) \ge \int_{\frac{x}{2}}^{x} \frac{L(t)}{t\log t}\,dt \ge L\left(\frac{x}{2}\right) \int_{\frac{x}{2}}^{x} \frac{dt}{t\log t} \gg \frac{L(x)}{\log x},$$

hence

$$\frac{L(x)}{\log^2 x} \ll \frac{I_L(x)}{\log x} = o\left(\frac{I_L(x)}{\log\log x}\right).$$

From (3.2) and (3.3) we have, for $p \le x/\xi(x)$,

(5.2) $$\sum_{m\le x/p} \frac{1}{\omega(mp)} = \frac{x}{p\log\log\frac{x}{p}} + O\left(\frac{x}{p(\log\log\frac{x}{p})^2}\right).$$

This gives

$$\sum\nolimits_1 = \sum_{p\le x/\xi(x)} L(p) \sum_{m\le x/p} \frac{1}{\omega(mp)}$$

(5.3) $$= x \sum_{p\le x/\xi(x)} \frac{L(p)}{p\log\log\frac{x}{p}} + O\left(x \sum_{p\le x/\xi(x)} \frac{L(p)}{p(\log\log\frac{x}{p})^2}\right).$$

Using the method of proof of Lemma 1, we obtain, for $y \le x/3$,

$$\sum_{p\le y} \frac{L(p)}{p\log\log\frac{x}{p}} = \int_{x_0}^{y} \frac{L(t)dt}{t\log t\log\log\frac{x}{t}} + O\left(\frac{L(y)}{\log^2 y}\right),$$

hence

$$\sum_{p\le x/\xi(x)} \frac{L(p)}{p\log\log\frac{x}{p}} = \int_{x_0}^{x/\xi(x)} \frac{L(t)dt}{t\log t\log\log\frac{x}{t}} + O\left(\frac{L(x/\xi(x))}{\log^2(x/\xi(x))}\right)$$
$$= \int_{x_0}^{x} \frac{L(t)dt}{t\log t\log\log\frac{x}{t}} + O\left(\frac{L(x)}{\log^2 x}\right),$$

since $L(x)$ is increasing and $\xi(x) \le \sqrt{x}$. Using the relation

$$\frac{1}{\log\log\frac{x}{t}} = \frac{1}{\log\log x}\left(1 + O\left(\frac{1}{\log\log\xi(x)}\right)\right) \quad \left(1 \le t \le \frac{x}{\xi(x)}\right),$$

and evaluating the O-term in (5.3) in a similar way, it follows that

(5.4) $$\sum\nolimits_1 = \frac{x}{\log\log x}\left(1 + O\left(\frac{1}{\log\log\xi(x)}\right)\right) I_L\left(\frac{x}{\xi(x)}\right) + O\left(\frac{xI_L(x)}{(\log\log x)^2}\right).$$

We have

$$0 \le I_L(x) - I_L\left(\frac{x}{\xi(x)}\right) = \int_{x/\xi(x)}^{x} \frac{L(t)}{t\log t}\,dt$$
$$\le L(x)\left(\log\log x - \log\frac{x}{\xi(x)}\right)$$
$$= L(x)\log\frac{1}{1-\frac{\log\xi(x)}{\log x}} \ll \frac{L(x)\log\xi(x)}{\log x}.$$

Thus from (5.4) and the upper bound on $\sum_2$, we obtain

(5.5) $$S(x) = (1+o(1))\frac{xI_L(x)}{\log\log x} + O\left(\frac{xL(x)\log\xi(x)}{\log x}\right).$$

Now we define

(5.6) $$\xi(x) = \exp\left\{\left(\frac{I_L(x)\log x}{L(x)\log\log x}\right)^{\frac{1}{2}}\right\}.$$

If (5.1) holds, then from (5.5) we obtain the assertion of the theorem, since $\sqrt{x} = o(x)$ as $x \to \infty$ and thus (5.6) gives

$$\frac{L(x)}{\log x}\log\xi(x) = \frac{L(x)}{\log x}\left(\frac{I_L(x)\log x}{L(x)\log\log x}\right)^{\frac{1}{2}} = o\left(\frac{L(x)}{\log x}\frac{I_L(x)\log x}{L(x)\log\log x}\right).$$

We have

$$I_L(x) = \int_{x_0}^{x} \frac{L(t)}{t\log t}\,dt \ll L(x)\int_{x_0}^{x} \frac{dt}{t\log t} \ll L(x)\log\log x,$$

which in view of (5.6) implies $\xi(x) \le \sqrt{x}$, as required. The main difficulty lies in showing that $\lim_{x\to\infty} \xi(x) = +\infty$, which will follow from

(5.7) $$\lim_{x\to\infty} \gamma_L(x) = +\infty, \quad \gamma_L(x) \overset{\text{def}}{=} \frac{I_L(x)\log x}{L(x)\log\log x}.$$

Here we distinguish between cases (i) and (ii). If (i) holds, then first we have that $L(x)\log\log x/\log x \to \infty$ and $I_L(x) \to \infty$ as $x \to \infty$. Using l'Hospital's rule and the definition of $\lambda_L(x)$, we obtain

$$\lim_{x\to\infty} \gamma_L(x) = \lim_{x\to\infty} \frac{I_L'(x)}{\left(L(x)\frac{\log\log x}{\log x}\right)'} = \lim_{x\to\infty} \frac{\log x}{(\lambda_L(x)-1)\log\log x + 1} = +\infty$$

precisely if $\lambda_L(x) = o\left(\frac{\log x}{\log\log x}\right)$, which is our hypothesis.

If (ii) holds, then for $t \ge x_1$ and some $c > 0$,

$$\frac{tL'(t)\log t}{L(t)} = \lambda_L(t) < c$$

and $L(x), I_L(x) \to \infty$ with x. Hence integrating the above inequality we obtain

$$L(x) - L(x_1) = \int_{x_1}^{x} L'(t)\,dt \le c\int_{x_1}^{x} \frac{L(t)}{t\log t}\,dt \ll I_L(x),$$

and (5.7) follows.

Having settled the cases (i) and (ii), we pass now to the case (iii). This time we choose

(5.8) $$\xi(x) = \frac{\log x}{(\log\log x)^2}.$$

Then we have, using Lemma 1 and Lemma 3,

$$\begin{aligned}\sum\nolimits_2 &\ll x \sum_{x/(\xi(x))<p\le x} \frac{L(p)}{p} \ll \xi(x) \sum_{x/\xi(x)<p\le x} L(p)\\ &= \xi(x)\left(\int_{x/\xi(x)}^{x} \frac{L(u)}{\log u}\,du + O\left(\frac{xL(x)}{\log^2 x}\right)\right)\\ &\ll \frac{x\xi(x)L(x)}{\log x} = \frac{xL(x)}{(\log\log x)^2} = o\left(\frac{xI_L(x)}{\log\log x}\right),\end{aligned}$$

where in the last step Lemma 5 (iii) was used, since now $L \in \mathcal{L}^*$. Hence we only need to prove that

(5.9) $$\sum\nolimits_1 = (1+o(1))\frac{xI_L(x)}{\log\log x}.$$

Using (5.2) and the method of proof of Lemma 1, we have

$$\begin{aligned}\sum\nolimits_1 &= (1+o(1))\frac{x}{\log\log x}\sum_{p\le x/\xi(x)} \frac{L(p)}{p}\\ &= (1+o(1))\frac{x}{\log\log x}\int_{x_0}^{x/\xi(x)} \frac{L(t)}{t\log t}\,dt + O\left(\frac{xL(x)}{\log^2 x\log\log x}\right)\\ &= (1+o(1))\frac{x}{\log\log x} I_L\left(\frac{x}{\xi(x)}\right) + o\left(\frac{xI_L(x)}{\log\log x}\right).\end{aligned}$$

By (5.8) and Lemma 5 (i) and (ii), we obtain $I_L\left(\frac{x}{\xi(x)}\right) \sim I_L(x)$, which establishes (5.9).

It remains yet to consider case (iv), that is, when, for some $d > 0$ and $x \ge x_1$,

(5.10) $$L(x) < \frac{1}{\log^d x}.$$

This implies that, as $y \to \infty$,

(5.11) $$I_L(y) \sim \sum_{p\le y} \frac{L(p)}{p} = C_1 + O\Big(\sum_{p>y} \frac{1}{p\log^d p}\Big) = C_1 + O\left(\frac{1}{\log^d y}\right),$$

where

$$C_1 = \sum_p \frac{L(p)}{p} > 0,$$

the summation being over all primes p. In this case we simply choose $\xi(x) = \sqrt{x}$. Using (5.10), we easily have

$$\sum\nolimits_2 \ll x \sum_{\sqrt{x}<p\le x} \frac{L(p)}{p} \ll x\int_{\sqrt{x}}^{\infty} \frac{dt}{t\log^{d+1} t} \ll \frac{x}{\log^d x} = o\left(\frac{xI_L(x)}{\log\log x}\right),$$

since $I_L(x)$ (by (5.11)) is bounded. To estimate $\sum_1$, we use the asymptotic development of $T(\frac{x}{p}, p)$ obtained in the proof of Theorem 1. We get, for any fixed integer $M \ge 1$,

$$\sum\nolimits_1 = x\sum_{p\le\sqrt{x}} \frac{L(p)}{p}\left\{\sum_{i=1}^{M} \frac{d_i}{\left(\log\log\frac{x}{p}\right)^i} + O\left(\frac{1}{(\log\log x)^{M+1}}\right)\right\}$$

with $d_1 = 1$. Replacing $\log\log\frac{x}{p}$ by $\log\log x$ and using (5.11), we obtain the assertion of Theorem 3 for case (iv). This ends the proof of Theorem 3.

Remark 3. Actually, using (5.11) and the method of proof of Theorem 1, what we obtain in case (iv) is the analogue of Theorem 1, namely

$$S(x) = \sum_{j=0}^{M} \frac{e_j x}{(\log\log x)^j} + O\left(\frac{x}{(\log\log x)^{M+2}}\right)$$

for any fixed integer $M \ge 0$ and effectively computable constants e_j ($e_0 = C_1 = \sum_p \frac{L(p)}{p}$).

Remark 4. If $\lambda_L(x)$ is not much smaller than $\log x$ (by definition of $L(x)$, $\lambda_L(x) = o(\log x)$ as $x \to \infty$), then Theorem 3 does not have to hold. This means that the restriction on $\lambda_L(x)$ in case (i) is a very reasonable one. Suppose that $\lambda_L(x)$ is increasing and $\lambda_L(x) > \frac{A\log x}{\log\log x}$, where $A > 0$ is a given constant. Then

$$S(x) \geq \sum_{\frac{x}{2}<p\leq x} L(p) \sum_{m\leq \frac{x}{p}} \frac{1}{\omega(pm)} \gg L\left(\frac{x}{2}\right) \sum_{\frac{x}{2}<p\leq x} 1 \gg \frac{xL(x)}{\log x}$$

by the prime number theorem. But $L'(t) = \frac{L(t)\lambda_L(t)}{t\log t}$ and our assumption on λ_L gives by integration

$$L(x) + O(1) \geq \int_{\sqrt{x}}^{x} \frac{L(t)\lambda_L(t)}{t\log t}\,dt \geq \frac{A\log\sqrt{x}}{\log\log\sqrt{x}}(I_L(x) - I_L(\sqrt{x})) \gg \frac{A\log x}{\log\log x} I_L(x),$$

which contradicts the assertion of Theorem 3 if A is large enough. Here we used the fact that $I_L(\sqrt{x}) = o(I_L(x))$ as $x \to \infty$. This follows by l'Hospital's rule and our assumption on $\lambda_L(x)$.

6. The probabilistic interpretation

As we mentioned in the introduction, for each choice of $f(n) = \sum_{p|n} f(p)$, the sum $S(x)$ given in (1.4) has a probabilistic interpretation: it represents the expected value of the sums $S_r(x)$ given by (1.2). In sections 3, 4 and 5 above, we obtained asymptotic estimates of $S(x)$ for different classes of slowly varying functions. Our results had the form

$$S(x) = (1 + o(1))\Lambda(x), \tag{6.1}$$

where $\Lambda(x)$ is a "nice" increasing function. As we shall see now, these estimates do indeed approximate "very well" the sums $S_r(x)$. If we denote by $N_Q(x)$ the number of sums $S_r(x)$ for which the relation

$$S_r(x) = (1 + o(1))\Lambda(x) \tag{6.2}$$

does not hold, then, as $x \to \infty$,

$$N_Q(x) = o(\omega(2)\ldots\omega([x])). \tag{6.3}$$

This will mean that, for almost all sums $S_r(x)$, the estimate (6.2) holds. In order to prove this, we will use the famous Chebyshev inequality from probability theory in the form stated below.

LEMMA 6. *Let $A_n = \{a_2, a_3, \ldots, a_{g(n)}\}$, $n \geq 2$, be a sequence of finite sets. For every $n \geq 2$, pick one member $r(n)$ of A_n at random with equal probabilities (i.e. $r(n) = a_j$ with probability $1/g(n)$), and set*

$$Q(x) = \sum_{2\leq n\leq x} r(n). \tag{6.4}$$

Then the number $N_Q(x)$ of sums (6.4) for which

$$|Q(x) - E| \geq U, \tag{6.5}$$

where

$$E = E(x) = \sum_{2\leq n\leq x} \frac{1}{g(n)} \sum_{j=2}^{g(n)} a_j$$

and

$$U = U(x) = o(E(x)) \tag{6.6}$$

satisfies

$$\frac{N_Q(x)}{g(2)g(3)\ldots g([x])} \leq \frac{V}{U^2}, \tag{6.7}$$

where

$$V = V(x) = \sum_{2\leq n\leq x} \frac{1}{g(n)} \sum_{j=2}^{g(n)} a_j^2 - \sum_{2\leq n\leq x} \left(\frac{1}{g(n)} \sum_{j=2}^{g(n)} a_j\right)^2.$$

Proof. For a proof of this lemma, see Galambos [5].

For our purposes we set $g(n) = \omega(n)$ in Lemma 6. Hence, in order to show that (6.2) holds for almost all sums $S_r(x)$, we need to choose $U = U(x)$ in such a way that condition (6.6) is satisfied and at the same time the right hand side of (6.7) tends to 0 as $x \to \infty$.

We treat separately the cases "$\rho \leq 0$" and "$\rho > 0$".

First observe that in all cases we have

$$V(x) \leq \sum_{2\leq n\leq x} \frac{1}{\omega(n)} \sum_{p|n} f^2(p). \tag{6.8}$$

We claim that a proper choice of $U(x)$ is

$$U(x) = E(x)^{\frac{1}{2}+\delta}, \tag{6.9}$$

where δ is any positive real number satisfying

$$\frac{1}{2} > \delta > \frac{\rho_*}{2\rho_* + 2}, \tag{6.10}$$

with $\rho_* = \max(0, \rho)$.

We start with the case "$\rho \leq 0$". As we have seen in Theorems 1 and 3, we have $E(x) \sim xM(x)$ for some $M \in \mathcal{L}$. On the other hand, it is easy to establish, using (6.8), that $V(x) \ll xN(x)$ for some $N \in \mathcal{L}$. Hence

$$\frac{V}{U^2} = \frac{V}{E^{1+2\delta}} \ll \frac{N(x)}{x^{2\delta}M(x)^{1+2\delta}},$$

which clearly tends to 0 as $x \to \infty$, because $\delta > 0$.

It remains to consider the case "$\rho > 0$". Theorem 3 ensures us that $E(x) \sim$ ${}^{\rho}M(x)$ for some $M \in \mathcal{L}$. On the other hand, one can easily obtain, using (6.8) estimate (4) of De Koninck - Mercier [4],

$$V(x) \leq \sum_{2 \leq n \leq x} f^2(P(n)) \sim \frac{x^{2\rho+1}\zeta(2\rho+1)}{2\rho+1} \frac{L^2(x)}{\log x} = x^{2\rho+1}N(x)$$

some $N \in \mathcal{L}$. It follows that

$$\frac{V}{U^2} = \frac{V}{E^{1+2\delta}} \ll x^{\rho-(2\rho+2)\delta} \frac{N(x)}{M(x)^{1+2\delta}},$$

hich also tends to 0 as $x \to \infty$, because of condition (6.10).

This proves our claim.

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Jean-Marie De Koninck
Département de mathématiques et statistique
Université Laval
Québec G1K 7P4
Canada

Aleksandar Ivić
Katedra Matematike
RGF-a Univerziteta u Beogradu
Djušina 7, 11000 Beograd
Yugoslavia