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Continuity module of the distribution of additive functions related to the largest prime factors of integers

By

J. M. DE KONINCK, I. KÁTAI and A. MERCIER

1. Introduction. For an integer $n > 1$, let $p(n)$ and $P(n)$ denote the smallest and the largest prime factor of n, respectively. The letters $c, c_1, c_2, \ldots$ denote suitable positive constants not necessarily the same at every occurence.

For some $\alpha > 0$ let

$$f(n) = \sum_{q \mid n} (\log q)^{\alpha}, \tag{1.1}$$

where the sum runs over the prime divisors of n,

$$v_x(n) \overset{\text{def}}{=} \frac{1}{(\log x)^{\alpha}} f(n), \tag{1.2}$$

$$T(n) \overset{\text{def}}{=} \frac{f(n)}{(\log P(n))^{\alpha}}. \tag{1.3}$$

In our previous paper [1] we proved that both $v_x(n)$ $(n \leqq x)$ and $T(n)$ have limit distributions. Let

$$F_x(y) = \frac{1}{x} \# \{n \leqq x \colon v_x(n) < y\}, \tag{1.4}$$

$$F(y) = \lim_{x \to \infty} F_x(y), \tag{1.5}$$

$$G_x(y) = \frac{1}{x} \# \{n \leqq x \colon T(n) < y\}, \tag{1.6}$$

$$G(y) = \lim_{x \to \infty} G_x(y). \tag{1.7}$$

Note that (1.5) and (1.7) hold only for points of continuity of the distribution functions; however, since F and G are continuous everywhere, this makes no difference. Let $\varrho(t)$ be defined for $t \geqq 1$ by

$$\lim_{x \to \infty} \frac{1}{x} \Psi(x, x^{1/t}) = \varrho(t), \tag{1.8}$$

where $\Psi(x, y)$ stands for the number of integers n up to x satisfying the condition $P(n) < y$.

It is known that ϱ is a decreasing function, that

$$\varrho(t) = \exp\left[-t\left(\log t + \log\log t - 1 - \frac{\log\log t}{\log t}\right) + O\left(\frac{1}{\log t}\right)\right], \tag{1.9}$$

as $t \to \infty$ and furthermore that

$$\Psi(x, x^{1/t}) = x\varrho(t) + O(x/\log x) \tag{1.10}$$

holds as $x \to \infty$ uniformly for all t varying in a bounded interval (see [1]).

The continuity modules of F and G, that is

$$Q_F(h) = \max_y (F(y+h) - F(y))$$

$$Q_G(h) = \max_y (G(y+h) - G(y))$$

will be treated here. We shall provide (mainly) upper bounds for $Q_F(h)$ and $Q_G(h)$, where $0 < h < 1$, for various ranges of α. Hence the results established in the following sections may be outlined as follows: let $0 < h < 1$, then

$$Q_F(h) \begin{cases} \leqq c\left(\log \frac{1}{h}\right)^{-1/2} & \text{if } 0 < \alpha < 1, \\ = 1 & \text{if } \alpha = 1, \\ \ll \left(\log \frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}} & \text{if } \alpha > 1, \end{cases}$$

and

$$Q_G(h) \begin{cases} < c\left(\log\log \frac{1}{h}\right)\left(\log \frac{1}{h}\right)^{-1/2} & \text{if } 0 < \alpha < 1, \\ = \log(1+h) & \text{if } \alpha = 1, \\ \leqq c\left(\log\log \frac{1}{h}\right)\left(\log \frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}} & \text{if } 1 < \alpha \leqq 2, \\ \leqq c h^{1/\alpha}\left(\log \frac{1}{h}\right)^{\alpha+1} & \text{if } \alpha > 2. \end{cases}$$

2. The case $\alpha = 1$. In this case, it is clear that $f(n) = \log n + o(\log n)$ holds on any set of integers having asymptotic density 1, whence we easily obtain that $F(1) = 0$, $F(1+0) = 1$, so F has a maximal jump in 1. Since ϱ is a continuous function, we get that

$$G(z) = 1 - \varrho(z) \quad \text{if } z \geqq 1. \tag{2.1}$$

Let $0 < h < 1$. Observe that $\varrho(z+h) - \varrho(z)$ is not greater than the limit density of the integers n up to x having at least one prime divisor in the interval $[x^{1/(z+h)}, x^{1/z}]$, and that this can be estimated from above by the limit bound of

$$\sum_{x^{1/(z+h)} \leqq p \leqq x^{1/z}} \frac{1}{p};$$

therefore

$$\varrho(z+h)-\varrho(z) \leqq \log\left(1+\frac{h}{z}\right) \tag{2.2}$$

holds for every $h \leqq 1$, $z \geqq 1$. Furthermore, as is well-known, in the interval $1 \leqq z \leqq 2$, $G(z) = \log z$, which gives immediately that

$$\max_z (G(z+h) - G(z)) = \log(1+h) \tag{2.3}$$

for every $h \leqq 1$.

3. Approximation of the distribution function *F*. Let $0 < \delta \leqq 1$,

$$v_{x,\delta}(n) = \frac{1}{(\log x)^\alpha} \sum_{\substack{q \mid n \\ q > x^\delta}} (\log q)^\alpha \tag{3.1}$$

and

$$S_{x,\delta}(n) = v_x(n) - v_{x,\delta}(n). \tag{3.2}$$

In [1] we noted that for every constant $a > 0$,

$$\frac{1}{x} \sum_{n \leqq x} e^{a v_x(n)} \leqq \prod_{p \leqq x} \left(1 + \frac{e^{a\left(\frac{\log p}{\log x}\right)^\alpha} - 1}{p}\right) \leqq c_1(a, \alpha) \tag{3.3}$$

where $c_1(a,\alpha)$ depends on a and α.

Assume that $0 \leqq a \leqq 1/\delta^\alpha$. Then

$$A_x \overset{\text{def}}{=} \frac{1}{x} \sum_{n \leqq x} e^{a S_{x,\delta}(n)} \leqq \prod_{p \leqq x^\delta} \left(1 + \frac{e^{a\left(\frac{\log p}{\log x}\right)^\alpha} - 1}{p}\right). \tag{3.4}$$

But the above product is less than

$$\exp\left(\sum_{p \leqq x^\delta} \frac{e^{a\left(\frac{\log p}{\log x}\right)^\alpha} - 1}{p}\right) \leqq \exp\left(2a \sum_{p \leqq x^\delta} \frac{1}{p}\left(\frac{\log p}{\log x}\right)^\alpha\right).$$

Therefore since

$$\sum_{p \leqq x^\delta} \frac{1}{p} (\log p)^\alpha = \frac{\delta^\alpha}{\alpha}(1 + o(1))(\log x)^\alpha,$$

we deduce that

$$A_x \leqq \exp(3a\delta^\alpha/\alpha) \tag{3.5}$$

if $x > c_2$. Hence we get immediately that for $x > x_0$,

$$\frac{1}{x} \# \{n \leqq x \colon S_{x,\delta}(n) \geqq K\delta^\alpha\} \leqq \exp(3/\alpha)\exp(-K/\alpha) \tag{3.6}$$

holds uniformly in K ($\geqq 1$).

Let $F_\delta(z)$ be the limit distribution of $v_{x,\delta}(n)$, the existence of which was proven in [1]. Since $v_{x,\delta}(n) \leqq v_x(n)$, therefore

(3.7) $$F_\delta(z) \geqq F(z)$$

holds for every z. Furthermore, $v_{x,\delta}(n) < z - K\delta^\alpha$, $v_x(n) \geqq z$ imply that $S_{x,\delta}(n) > K\delta^\alpha$, and so

(3.8) $$F_\delta(z - K\delta^\alpha) \leqq F(z) + \exp\left(\frac{3}{\alpha}\right)\exp\left(-\frac{K}{\alpha}\right).$$

Let

(3.9) $$Q_\delta(h) \overset{\text{def}}{=} \max_z (F_\delta(z+h) - F_\delta(z))$$

and

(3.10) $$Q_F(h) \overset{\text{def}}{=} \max_z (F(z+h) - F(z)).$$

If we choose now $\delta = h^{1/\alpha}$, from (3.7), (3.8) we obtain that

$$Q_F(h) \leqq \max_z (F_\delta(z+h) - F_\delta(z - K\delta^\alpha)) + \exp(3/\alpha)\exp(-K/\alpha),$$

whence

(3.11) $$Q_F(\delta^\alpha) \leqq (K+2)\, Q_\delta(\delta^\alpha) + \exp(3/\alpha)\exp(-K/\alpha).$$

We can deduce similarly that

(3.12) $$Q_\delta(\delta^\alpha) \leqq (K+2)\, Q_F(\delta^\alpha) + \exp(3/\alpha)\exp(-K/\alpha).$$

4. Estimation of $Q_F(h)$ in the case $\alpha > 1$. Let

(4.1) $$E(y) \overset{\text{def}}{=} F_\delta(y+h) - F_\delta(y),$$

where $h = \delta^\alpha$. Choose a fixed $y > \delta^\alpha$. Let $\mathcal{T}_x$ be the set of integers $n \leqq x$ for which $v_{x,\delta}(n) \in [y, y+h]$. It is clear that

$$\frac{\operatorname{card}(\mathcal{T}_x)}{x} \to E(y) \qquad (x \to \infty).$$

Let $\mathcal{T}_x^* \subseteqq \mathcal{T}_x$ be the subset of those integers n for which $p^2 \nmid n$ if $p > x^\delta$. Then $\operatorname{card}(\mathcal{T}_x \setminus \mathcal{T}_x^*) = o(x)$ $(x \to \infty)$. For a general n, let $p_1 > p_2 > \cdots > p_r$ be the set of all prime divisors greater than x^δ. We shall write

$$n_1 = p_1 p_2 \cdots p_r, \qquad M = p_2 \cdots p_r.$$

Let us estimate $\operatorname{card}(\mathcal{T}_x^*)$. Since $y > \delta^\alpha$, then clearly $n_1 = 1$ cannot occur. Now let $\xi > \delta$ be fixed. Then

(4.2) $$\operatorname{card}(\mathcal{T}_x^*) \leqq \sum \Psi\left(\frac{x}{Mp_1}, x^\delta\right) + \Psi(x, x^\xi) = \sum + \Psi(x, x^\xi),$$

where in $\sum$ we sum only over those $n_1 \in \mathscr{T}_x^*$ for which $p_1 > x^{\xi}$. We shall now make use of the inequality

$$\Psi(x, x^{\lambda}) \ll x \exp\left(-\frac{c}{\lambda}\right) \tag{4.3}$$

valid uniformly in the range $0 < \lambda \leqq 1$.

To estimate $\sum$ on the right hand side of (4.2), we shall distinguish between two cases, namely:

(A) $y > 8\xi^{\alpha}$,

(B) $y < 8\xi^{\alpha}$.

Case (A). Since for $n_1 \in \mathscr{T}_x^*$, we have

$$v_{x,\delta}(n_1) = \left(\frac{\log p_1}{\log x}\right)^{\alpha} + v_{x,\delta}(M) \in [y, y+h), \tag{4.4}$$

it follows that, for a given M, there exists $p_1 > x^{\xi}$ only if $v_{x,\delta}(M) < y + h - \xi^{\alpha}$. We shall write $\sum$ as $\sum_1 + \sum_2$, where in $\sum_1$, we have the restriction $v_{x,\delta}(M) \leqq y - 4\xi^{\alpha}$, while in $\sum_2$, we have the restriction $v_{x,\delta}(M) > y - 4\xi^{\alpha}$. It is clear that

$$\sum\nolimits_1 \leqq x \sum_{\substack{p_1 > x^{\xi} \\ n_1 \in \mathscr{T}_x^*}} \frac{1}{M p_1}.$$

For a fixed M, p_1 satisfies the inequalities

$$y - v_{x,\delta}(M) < \left(\frac{\log p_1}{\log x}\right)^{\alpha} < y + h - v_{x,\delta}(M);$$

this implies that

$$\sum \frac{1}{p_1} \ll \frac{1}{\alpha} \log \frac{y - v_{x,\delta}(M) + h}{y - v_{x,\delta}(M)} \ll \frac{h}{y - v_{x,\delta}(M)}.$$

Hence we obtain that

$$\sum\nolimits_1 \ll x h \sum \frac{1}{M(y - v_{x,\delta}(M))}.$$

Finally, since $y - v_{x,\delta}(M) \geqq 4\xi^{\alpha}$ and

$$\sum \frac{1}{M} \ll \prod_{x^{\delta} \leqq q \leqq x} \left(1 + \frac{1}{q}\right) \ll \frac{1}{\delta},$$

we obtain that

$$\sum\nolimits_1 \ll \frac{x \delta^{\alpha - 1}}{\xi^{\alpha}}.$$

On the other hand, if $v_{x,\delta}(M) > y - 4\xi^{\alpha}$ and $v_{x,\delta}(n_1) < y + h$, then

$$\left(\frac{\log p_1}{\log x}\right)^{\alpha} < h + 4\xi^{\alpha} \quad \text{and} \quad \log p_1 < \beta \log x,$$

with

(4.5) $$\beta = (h + 4\xi^\alpha)^{1/\alpha}.$$

But the number of integers $n \leqq x$ with $P(n) = p_1 \leqq x^\beta$ is precisely $\Psi(x, x^\beta)$; consequently

$$\sum_2 \ll x \exp\left(-\frac{c}{\beta}\right).$$

Therefore, in case (A), we have

(4.6) $$E(y) \ll \frac{\delta^{\alpha-1}}{\xi^\alpha} + \exp\left(-\frac{c}{\beta}\right)$$

with β as in (4.5).

Case (B). Since $(\log p_1/\log x)^\alpha < 8\xi^\alpha + h$, it follows that $\log p_1 < \gamma \log x$ with $\gamma = (8\xi^\alpha + h)^{1/\alpha}$ and hence that

$$\sum \ll \Psi(x, x^\gamma) \ll x \exp\left(-\frac{c}{\gamma}\right).$$

Since $\xi < \gamma$, we have

(4.7) $$E(y) \ll x \exp\left(-\frac{c}{\gamma}\right).$$

Now let $y \leqq \delta^\alpha$. Then $E(y) \leqq F_\delta(2\delta^\alpha) \leqq \xi(1/2^{1/\alpha}\delta)$, and so by (4.3) one has that $E(y) \ll \delta^\alpha$.

This in turn implies that

(4.8) $$Q_\delta(\delta^\alpha) \ll \frac{\delta^{\alpha-1}}{\xi^\alpha} + \exp\left(-\frac{c}{\gamma}\right).$$

Set $\gamma = \frac{c}{\alpha}\left(\log\frac{1}{\delta}\right)^{-1}$ and let ξ be computed from the equation $\gamma = (8\xi^\alpha + h)^{1/\alpha}$. Using (4.8), we easily get that

(4.9) $$Q_\delta(\delta^\alpha) \ll \left(\log\frac{1}{\delta}\right)^\alpha \delta^{\alpha-1}.$$

Let us choose now $K = c_1 \log(1/\delta)$ and apply (3.11). Then we have

$$G_F(\delta^\alpha) \ll \left(\log\frac{1}{\delta}\right)^{\alpha+1} \delta^{\alpha-1}.$$

This means that the following theorem is true.

Theorem 1. *If* $\alpha > 1$, *then*

$$Q_F(h) \ll \left(\log\frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}}.$$

5. Estimation of $Q_F(h)$ in the case $\alpha < 1$. A general theorem of I. Z. Ruzsa immediately implies the following

Theorem 2 (Ruzsa [2]). *Let $\alpha < 1$. Then*

$$Q_F(h) \leqq c(\log 1/h)^{-\frac{1}{2}}. \tag{5.1}$$

Ruzsa presented his result in the following form:

Let f be an additive function and let

$$Q_1(x) = \sup_a \frac{1}{x} \# \{n \leqq x \colon f(n) \in [a, a+1]\},$$

$$W(x, \lambda) = \sum_{p \leqq x} \frac{1}{p} \min(1, (f(p) - \lambda \log p)^2),$$

$$W(x) = \min_\lambda (\lambda^2 + W(x, \lambda)).$$

Then

$$Q_1(x) \ll (W(x))^{-\frac{1}{2}}.$$

To see how Theorem 2 follows from the above, we consider the function

$$f(n) \stackrel{\text{def}}{=} \frac{1}{h} \sum_{q \mid n} \left(\frac{\log q}{\log x}\right)^\alpha,$$

and let $h = \delta^\alpha$. We have

$$W(x, 0) = \sum_{p \leqq x^\delta} \frac{1}{p} \left(\frac{\log p}{\delta \log x}\right)^{2\alpha} + \sum_{x^\delta < p < x} \frac{1}{p} = \log \frac{1}{\delta} + O(1).$$

It is enough to see that $\lambda^2 + W(x, \lambda) \gg W(x, 0)$ for every real number λ. Let κ be defined by the relation $\lambda = \kappa/(\delta \log x)$ and set $\eta_u = (\log u)/(\delta \log x)$. Then

$$W(x, \lambda) = \sum_p \frac{1}{p} \min\{1, (\eta_p^\alpha - \kappa \eta_p)^2\}.$$

If $\kappa \geqq 1$, then $\kappa \eta_p - \eta_p^\alpha \geqq c - c^\alpha$ for $p \geqq x^{c\delta}$ and so $W(x, \lambda) \gg \log 1/\delta$. Assume now that $0 < \kappa \leqq 1$. Then

$$W(x, \lambda) = \sum_{x^\delta < p < x} \frac{1}{p} - \sum \frac{1}{p},$$

where the second sum runs over the primes p such that $x^\delta < p < x$ and for which $|\eta_p^\alpha - \kappa \eta_p| < 1$. Let $\eta^{(1)} < \eta^{(0)}$ be defined as the solutions of the equations

$$\begin{cases} \eta^{(1)\alpha} - \kappa \eta^{(1)} = 1, \\ \eta^{(0)\alpha} - \kappa \eta^{(0)} = -1. \end{cases}$$

It is clear that

$$\sum_{p \colon \eta^{(1)} < \eta_p < \eta^{(0)}} \frac{1}{p} < \log \frac{\eta^{(0)}}{\eta^{(1)}} + O\left(\frac{1}{\log x}\right) < c_1,$$

and that $|\eta_p^\alpha - \kappa\eta_p| \geqq 1$ if $p \geqq x^\delta$ and $\eta_p \notin (\eta^{(1)}, \eta^{(0)})$. Thus we have

$$W(x, \lambda) \gg \log 1/\delta.$$

This ends the proof of Theorem 2.

Most likely, Theorem 2 is far from being sharp, but we are unable to prove a better one.

6. Estimation of $Q_G(h)$ in the case $\alpha < 1$. Let $y \geqq 1$, $h > 0$ be given. Then the density of the integers n satisfying $y \leqq T(n) < y + h$ can be estimated by $\frac{1}{x}\operatorname{card}(\mathcal{T}_x)$, where $\mathcal{T}_x$ is the set of those integers $n \leqq x$ for which

$$\frac{f(n)}{(\log P(n))^\alpha} \in [y, y+h) \tag{6.1}$$

holds. It is clear that $\frac{1}{x}\operatorname{card}(\mathcal{T}_x) \to G(y+h) - G(y)$. We shall give an upper estimate for the number of integers $n \leqq x$ satisfying (6.1). Let us write each $n \in \mathcal{T}_x$ as $n = mP$, where $P = P(n)$ is the largest prime divisor of n. Let λ_1, λ_2 be small positive numbers. The number of those elements n of $\mathcal{T}_x$ for which $P(n) \leqq x^{\lambda_1}$ is not greater than $\Psi(x, x^{\lambda_1})$, while the number of those for which $P(n) > x^{1-\lambda_2}$ is at most $x \sum_{x^{1-\lambda_2} < p \leqq x} 1/p$. Thus, with the exception of at most $\lambda_2 x + x e^{-c/\lambda_1} + o(x)$ integers, we may assume that $x^{\lambda_1} < P(n) \leqq x^{1-\lambda_2}$. The number of integers $n \leqq x$ for which $P^2(n) \mid n$ and $P(n) > x^{\lambda_1}$ is at most $O(x^{1-\lambda_1})$. For the others, we have $f(n) = f(P(n)) + f(m)$; for each fixed P, (6.1) can be written as

$$\frac{f(m)}{(\log P)^\alpha} \in [y-1, y-1+h),$$

and so

$$\frac{f(m)}{(\log(x/P))^\alpha} \in \left[\frac{(y-1)(\log P)^\alpha}{(\log(x/P))^\alpha}, \frac{(y+h-1)(\log P)^\alpha}{(\log(x/P))^\alpha}\right). \tag{6.2}$$

If $n = Pm$, $P(n) = P$, $n \in \mathcal{T}_x$, then $m \leqq x/P$ and $P(m) < P$. Let $\mathcal{T}_{x,P}$ be the set of the integers m, for which $m \leqq x/P$ and (6.2) holds. We keep in $\mathcal{T}_{x,P}$ the elements with $P(m) \geqq P$ as well. Then, it is clear that

$$\operatorname{card}(\mathcal{T}_x) \leqq \sum_{x^{\lambda_1} < P \leqq x^{1-\lambda_2}} \operatorname{card}(\mathcal{T}_{x,P}) + \lambda_2 x + x e^{-c/\lambda_1} + o(x). \tag{6.3}$$

The length of the interval on the right hand side of (6.2) is $h(\log P)^\alpha/(\log(x/P))^\alpha$; therefore,

$$\operatorname{card}(\mathcal{T}_{x,P}) \leqq \frac{x}{P} Q_F\left(\frac{h(\log P)^\alpha}{(\log(x/P))^\alpha}\right) + o(x/P), \tag{6.4}$$

uniformly for $P \in [x^{\lambda_1}, x^{1-\lambda_2}]$. We are now in a position to apply Theorem 2.

Summing up for $P \leqq \sqrt{x}$ and observing that $\dfrac{(\log P)^\alpha}{(\log(x/P))^\alpha} \leqq 1$, we have

$$\sum_{P \leqq \sqrt{x}} \operatorname{card}(\mathcal{T}_{x,P}) \leqq Q_F(h)\, x \sum_{x^{\lambda_1} \leqq P \leqq x^{1/2}} \frac{1}{P} + o(x) \leqq Q_F(h)\, x \log \frac{1}{\lambda_1} + o(x).$$

We now choose $\lambda_1 = (c \log\log 1/h)^{-1}$ and $\lambda_2 = (\log 1/h)^{-\frac{1}{2}}$ and further proceed to sum over $P \in [x^{1/2}, x^{1-\lambda_2}]$. Then, for every P,

$$h \frac{(\log P)^\alpha}{(\log(x/P))^\alpha} < h(\log 1/h)^{\alpha/2},$$

$$Q_F\left(h \frac{(\log P)^\alpha}{(\log(x/P))^\alpha}\right) \ll (\log\log 1/h)(\log 1/h)^{-1/2}.$$

Consequently

$$\sum_{\sqrt{x} \leqq P < x^{1-\lambda_2}} \operatorname{card}(\mathcal{T}_{x,P}) \ll x\left(\log\log \frac{1}{h}\right)\left(\log \frac{1}{h}\right)^{-\frac{1}{2}}.$$

Collecting our inequalities we immediately obtain

Theorem 3. *If $\alpha < 1$, then for every h, $0 < h < 1$,*

$$Q_G(h) < C\left(\log\log \frac{1}{h}\right)\left(\log \frac{1}{h}\right)^{-1/2}.$$

7. Estimation of $G(1 + \omega^\alpha)$ for small ω. For an integer n let $Q(n)$ denote its second largest prime factor. Since $T(n) < \omega^\alpha$ implies that $Q(n) < P(n)^\omega$, it follows that $G(1 + \omega^\alpha)$ is not greater than the upper density of the integers n satisfying $Q(n) < P(n)^\omega$. Observing that the number of integers $n \leqq x$ satisfying this condition is at most

$$(7.1) \qquad \sum \sum_{Q < P^\omega} \Psi\left(\frac{x}{PQ}, Q\right).$$

By a simple application of the inequality

$$\Psi(x, y) < x \exp\left(-c \frac{\log x}{\log y}\right),$$

we easily get that (4.1) is less than $c_1 x \omega$. This implies that

$$(7.2) \qquad G(1 + \omega^\alpha) \leqq c_1 \omega.$$

Let us now consider all those integers n up to x which can be written as $n = mp$, where $m \leqq x^\omega$, p is a prime larger than $\sqrt{x}$. For such an n,

$$(7.3) \qquad T(n) = 1 + \frac{1}{(\log p)^\alpha} \sum_{q \mid m} (\log q)^\alpha \leqq 1 + \frac{2^\alpha (\log m)^\alpha}{(\log x)^\alpha} \frac{1}{(\log m)^\alpha} \sum_{q \mid m} (\log q)^\alpha$$

$$= 1 + \frac{2^\alpha (\log m)^\alpha}{(\log x)^\alpha} s(m),$$

say. For every m we have at least $c_1 x/m \log x$ distinct n's. Let K be a positive number. If $s(m) = K$, then for every $n = mp$ we have

$$T(n) \leqq 1 + \frac{2^\alpha (\log m)^\alpha}{(\log x)^\alpha} K \leqq 1 + 2^\alpha K \omega^\alpha.$$

Let $K = 1$. The density of the integers m satisfying $s(m) \leqq 1$ is positive; therefore for every large M, in the interval $[2^M, 2^{M+1}]$ there exists at least $c_2 2^M$ distinct m's such that $s(m) \leqq 1$. This implies rapidly that $T(n) \leqq 1 + 2^\alpha \omega$ holds for at least

$$c_1 c_2 \frac{x}{\log x} \sum_{m \leqq x^\omega} \frac{1}{m} \geqq \frac{1}{2} c_1 c_2 x \omega$$

distinct integers. Thus $G(1 + 2^\alpha \omega^\alpha) \geqq c_3 \omega$. We have thus proved the following

Theorem 4. *If* $\alpha > 0$, *then*

$$c_1 \omega < G(1 + \omega^\alpha) < c_2 \omega$$

with suitable positive constants c_1, c_2 *which may depend on* α.

8. Estimation of $Q_G(h)$ in the case $\alpha > 1$. Let $y \geqq 1$ and $0 < h < 1$ be fixed. We argue similarly as in Section 6. For $E(y, h) = G(y + h) - G(y)$, we have

$$xE(y,h) \leqq x \exp\left(-\frac{c}{\lambda_1}\right) + \lambda_2 x + \sum_{p^{\lambda_1} < P < \sqrt{x}} \operatorname{card}(\mathscr{T}_{x,P}) + \sum_{\sqrt{x} \leqq P < x^{1-\lambda_2}} \operatorname{card}(\mathscr{T}_{x,P}). \tag{8.1}$$

It is clear that

$$\operatorname{card}(\mathscr{T}_{x,P}) \leqq \frac{x}{P} Q_F(h) + o\left(\frac{x}{P}\right) \quad \text{if } P < \sqrt{x}; \tag{8.2}$$

therefore the first sum on the right hand side of (8.1) is bounded by

$$x Q_F(h) \log \frac{1}{\lambda_1} + o(x). \tag{8.3}$$

Let $y > 1$. Since $f(m) / \left(\log \frac{x}{P}\right)^\alpha$ in (6.2) is smaller than 1, it follows that when estimating the second sum on the right hand side of (8.1), we may assume that

$$\frac{\log P}{\log(x/P)} \leqq \frac{1}{(y-1)^{1/\alpha}},$$

hence that

$$\frac{\log P}{\log x} \leqq \frac{1}{1 + (y-1)^{1/\alpha}} \stackrel{\text{def}}{=} \theta. \tag{8.4}$$

From Theorem 1, we have

$$Q_F\left(h\left(\frac{\log P}{\log(x/P)}\right)\right) \ll \left(\log\frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}}\left(\frac{\log P}{\log x/P}\right)^{\alpha-1},$$

and so

$$\text{(8.5)} \qquad \sum_{P \geqq \sqrt{x}} \operatorname{card}(\mathcal{T}_{x,P}) \leqq x\left(\log\frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}} \sum \frac{1}{P}\left(\frac{\log P}{\log x/P}\right)^{\alpha-1} + o\left(x \sum_{\sqrt{x} \leqq P < x} \frac{1}{P}\right),$$

where we sum only over those primes P for which (8.4) holds. Let $\sum_{\theta,\alpha}$ denote the first sum on the right hand side of (8.5). From the prime number theorem we get immediately that

$$\text{(8.6)} \qquad \sum_{\theta,\alpha} \leqq \begin{cases} c_1 & \text{if } \alpha \leqq 2, \\ c_2(1-\theta)^{2-\alpha} & \text{if } \alpha > 2, \end{cases}$$

where c_1, c_2 depend on α, but not on y.

Assume that $\alpha \leqq 2$. Choosing $\lambda_1 = c/(\log 1/h)$ in (8.3) and using (8.5), (8.6) and further setting $\lambda_2 = 0$ in (8.1) and taking into account Theorem 4 for the case $y = 1$, we obtain

Theorem 5. *If* $1 < \alpha \leqq 2$, *then*

$$Q_G(h) \leqq c\left(\log\log\frac{1}{h}\right)\left(\log\frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}}.$$

Assume now that $\alpha > 2$. Let $\lambda_1 = c/\log(1/h)$ and $\lambda_2 = 0$ as earlier. Assume first that $y - 1 \geqq h$. Then by (8.6), we obtain that (8.5) is bounded by

$$\ll x\left(\log\frac{1}{h}\right)^{\alpha+1} h^{1-\frac{1}{\alpha}}(y-1)^{\frac{2}{\alpha}-1} + o(x) \ll x h^{1/\alpha}(\log 1/h)^{\alpha+1} + o(x).$$

Hence we obtain that

$$E(y) \ll h^{1/\alpha}(\log 1/h)^{\alpha+1}$$

uniformly in y as $y - 1 \geqq h$. But Theorem 3 gives that the same is true if $1 \leqq y \leqq 1 + h$. Hence we have

Theorem 6. *If* $\alpha > 2$, *then*

$$c_1 h^{1/\alpha} \leqq Q_G(h) \leqq c_2 h^{1/\alpha}(\log 1/h)^{\alpha+1},$$

for $0 < h < 1$, *with suitable positive constants that may depend on* α.

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Anschriften der Autoren:

Jean-Marie De Koninck
Département de Mathématiques
Université Laval
Québec, G1K 7P4
Canada

Imre Kátai
Eötvös Loránd University
Computer Center
1117 Budapest, Bogdánfy u. 10/B
Hungary

Armel Mercier
Département de Mathématiques
Université du Québec
Chicoutimi, G7H 2B1
Canada