Arch. Math., Vol. 52, 440–448 (1989)

0003-889X/89/5205-0440 $ 3.30/0

The average prime divisor of an integer in short intervals

By

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1. Introduction. In [1] we defined two functions which may be called the average prime divisor of an integer. They are, for $n \geqq 2$,

$$P_*(n) = \frac{\beta(n)}{\omega(n)}, \quad P^*(n) = \frac{B(n)}{\Omega(n)}, \quad \beta(n) = \sum_{p \mid n} p, \quad B(n) = \sum_{p^\alpha \| n} \alpha p,$$

and as usual $\omega(n)$ is the number of distinct prime factors of n, $\Omega(n)$ is the total number of prime factors of n, $p^\alpha \| n$ means that p^α (p prime) exactly divides n. It was shown in [1] that, for each fixed natural number m, there exist computable constants $e_1, e_2, \ldots, e_m$, $f_1, f_2, \ldots, f_m$, $0 < f_1 < e_1$, such that

$$(1.1) \qquad \sum_{2 \leqq n \leqq x} P_*(n) = x^2 \left(\frac{e_1}{\log x} + \frac{e_2}{\log^2 x} + \cdots + \frac{e_m}{\log^m x} + O\left(\frac{1}{\log^{m+1} x} \right) \right),$$

$$(1.2) \qquad \sum_{2 \leqq n \leqq x} P^*(n) = x^2 \left(\frac{f_1}{\log x} + \frac{f_2}{\log^2 x} + \cdots + \frac{f_m}{\log^m x} + O\left(\frac{1}{\log^{m+1} x} \right) \right).$$

We also proved that an asymptotic formula similar to (1.1) and (1.2) holds also for sums of $\beta(n)$, $B(n)$ and $P(n)$ (the largest prime factor of $n \geqq 2$). These questions were considered in a general setting by De Koninck and Mercier [2], where under suitable conditions asymptotic relations of the form

$$(1.3) \qquad \sum_{2 \leqq n \leqq x} f(P(n)) = (1 + o(1)) \sum_{2 \leqq n \leqq x} f(n) \quad (x \to \infty)$$

are established, and in fact both sums in (1.3) are evaluated asymptotically. Here $f(n)$ denotes a "large", strongly additive arithmetic function defined by the relation

$$(1.4) \qquad f(n) = \sum_{p \mid n} p^\varrho L(p)$$

for some $\varrho > 0$, and where $L(x)$ is a slowly oscillating (or slowly varying) function. Slowly oscillating functions (see E. Seneta [8] for a comprehensive account) are continuous for $x \geqq x_0$, and satisfy $\lim_{x \to \infty} L(Kx)/L(x) = 1$ for any fixed $K > 0$. For our purposes (similarly as in [2]), we shall consider positive slowly oscillating functions $L(x)$ which are defined for $x \geqq 2$ and may be written, for $x \geqq x_0 \geqq 2$, as

$$L(x) = K \exp\left(\int_{x_0}^{x} \eta(t)\,\frac{dt}{t}\right), \quad \lim_{t\to\infty} \eta(t) = 0, \quad K > 0. \tag{1.5}$$

Indeed, it was shown already by J. Karamata [5], who founded the theory of slowly oscillating functions, that a general slowly oscillating function $L(x)$ has a representation of the form (1.5) where K is not a constant but a function $K = K(x)$ which tends to a finite limit as $x \to \infty$. Thus every slowly oscillating function is asymptotic to a function of the form (1.5), and the latter have the advantage of being differentiable for $x \geqq x_0$. In analogy with $\omega(n)$, $\Omega(n)$ and $\beta(n)$, $B(n)$ we define also the additive function $F(n)$, which may be thought of as the companion function of $f(n)$. This is, for $n \geqq 2$,

$$F(n) = \sum_{p^\alpha \| n} \alpha p^\varrho L(p) \quad (\varrho > 0). \tag{1.6}$$

Our aim is to investigate the behaviour of $f(n)$, $F(n)$ and $P_*(n)$, $P^*(n)$ in "short" intervals $[x, x+h]$, where "short" means that $h = o(x)$ as $x \to \infty$. It will turn out that these problems are in an intrinsic way connected to the problem of the range of validity of the asymptotic relation

$$\pi(x+h) - \pi(x) = (1 + o(1))\,\frac{h}{\log x}, \tag{1.7}$$

where as usual $\pi(x) = \sum_{p \leqq x} 1$ is the number of primes not exceeding x. This is one of the most famous problems of prime number theory, and it is known e.g. that on the Lindelöf hypothesis ($\zeta(\frac{1}{2} + it) \ll |t|^\varepsilon$) (1.7) holds for $h = x^{1/2+\varepsilon}$. We are interested in unconditional results, and thus we can use (1.7) in the range

$$x^{7/12} \log^{22} x \leqq h \leqq x. \tag{1.8}$$

This was proved by A. Ivić [3] (see also [4], Ch. 12), and in fact (1.8) was shown to hold with $h \geqq x^{7/12} \log^{22-\delta} x$ with some $\delta > 0$. The constant $\frac{7}{12}$ comes from $\frac{7}{12} = 1 - \frac{1}{C}$, where $C = \frac{12}{5}$ is the currently best known constant (see Ch. 11 of [4]) for which the zero-density estimate $N(\sigma, T) \ll T^{C(1-\sigma)} \log^D T$ $(D \geqq 0)$ holds. Here as usual $N(\sigma, T)$ denotes the number of zeros $\varrho = \beta + i\gamma$ of $\zeta(s)$ for which $\beta \geqq \sigma \geqq \frac{1}{2}$, $|\gamma| \leqq T$. It is not hard to guess the behaviour of $P_*(n)$ and $P^*(n)$ in short intervals, although the difficulty in dealing with them lies in the fact that they are neither additive nor multiplicative. Indeed, from (1.1) we have by the mean value theorem

$$\sum_{x<n\leqq x+h} P_*(n) = e_1\,\frac{(x+h)^2}{\log(x+h)} - e_1\,\frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right) = (2e_1 + o(1))\,\frac{hx}{\log x}$$

for

$$\frac{x\phi(x)}{\log x} \leqq h \leqq o(x), \quad \text{if } \lim_{x\to\infty} \phi(x) = \infty, \quad \phi(x) = o(\log x), \quad (x \to \infty)$$

since $(x^2/\log x)' = 2x/\log x - x/\log^2 x$. Naturally, the above range of h is very poor, and a similar type of result could be obtained for $f(n)$ and $F(n)$ in short intervals by the main term for $\sum_{n \leqq x} f(n)$ evaluated in [2] (the methods of [2] will also work for $F(n)$ at the cost of some technical complications). It is the aim of this paper to obtain much better ranges for h, by connecting these problems to (1.7) and (1.8). We shall prove

Theorem 1. *Let $f(n)$ and $F(n)$ be defined by* (1.4) *and* (1.6) *respectively, with $L(x)$ given by* (1.5). *Then for $x^{7/12}\log^{22} x \leqq h \leqq o(x)$ we have, as $x \to \infty$,*

$$\sum_{x<n\leqq x+h} f(n) = (\zeta(1+\varrho) + o(1))\,\frac{h x^{\varrho} L(x)}{\log x} \tag{1.9}$$

and

$$\sum_{x<n\leqq x+h} F(n) = (\zeta(1+\varrho) + o(1))\,\frac{h x^{\varrho} L(x)}{\log x}. \tag{1.10}$$

Note that for $\varrho = 1$, $L(x) = 1$ we obtain $f(n) = \beta(n)$, $F(n) = B(n)$, hence Theorem 1 gives the asymptotic behaviour of $\sum_{x<n\leqq x+h} \beta(n)$ and $\sum_{x<n\leqq x+h} B(n)$.

Our second result concerns the behaviour of $P_*(n)$ and $P^*(n)$ in short intervals. Our method of approach certainly makes it possible to deal with more general arithmetic functions than $P_*(n)$ and $P^*(n)$, but we wanted to single out these two because of their arithmetic significance, since either of them may be thought of as the average prime divisor of n.

Theorem 2. *For $x^{7/12}\log^{22} x \leqq h \leqq o(x)$ we have, as $x \to \infty$,*

$$\sum_{x<n\leqq x+h} P_*(n) = (2e_1 + o(1))\,\frac{hx}{\log x} \tag{1.11}$$

and

$$\sum_{x<n\leqq x+h} P^*(n) = (2f_1 + o(1))\,\frac{hx}{\log x}, \tag{1.12}$$

where $0 < f_1 < e_1$ are the constants appearing in (1.1) *and* (1.2), *respectively.*

2. Proof of Theorem 1. We shall give only the proof of the somewhat more complicated relation (1.10), since both proofs are similar. Note that if $R(x) = x^{\varrho} L(x)$, then

$$R'(x) = \varrho x^{\varrho-1} L(x) + x^{\varrho} L'(x) = x^{\varrho-1} L(x)\ (\varrho + \eta(x)) > 0$$

for $x \geqq x_1$, since $\lim_{x\to\infty} \eta(x) = 0$. Hence $R(x)$ is increasing for $x \geqq x_1$ and

$$\begin{aligned}\sum_{x<n\leqq x+h} F(n) &= \sum_{x<n\leqq x+h,\, P(n)\leqq \sqrt{x}} F(n) + \sum_{x<n\leqq x+h,\, P(n)>\sqrt{x}} F(n)\\ &= \sum_{x<n\leqq x+h,\, P(n)>\sqrt{x}} F(n) + O(x^{\varrho/2+\varepsilon} h),\end{aligned}$$

since for $x_1 < P(n) \leqq \sqrt{x}$

$$F(n) \leqq \{P^{\varrho}(n)\, L(P(n)) + O(1)\} \sum_{p^{\alpha} \,\|\, n} \alpha = \{P^{\varrho}(n)\, L(P(n)) + O(1)\}\, \Omega(n) \ll x^{\varrho/2+\varepsilon},$$

because trivially $\Omega(n) \ll \log n$, and it follows from (1.5) that $L(x) \ll x^{\varepsilon}$ for any $\varepsilon > 0$. If q denotes primes, then we have

$$\begin{aligned}\sum_{x<n\leqq x+h,\, P(n)>\sqrt{x}} F(n) &= \sum_{x<n\leqq x+h,\, P(n)>\sqrt{x}} P^{\varrho}(n) L(P(n)) + \sum_{\substack{x<n\leqq x+h \\ P(n)>\sqrt{x}}} \sum_{\substack{q^k \| n \\ q<P(n)}} k q^{\varrho} L(q) \\ &= \sum_{x<n\leqq x+h,\, P(n)>\sqrt{x}} P^{\varrho}(n)\, L(P(n)) + O(h x^{\varrho/2+\varepsilon}),\end{aligned}$$

since $q\sqrt{x} \leqq q^k P(n) \leqq n \leqq x + h$ implies $q \leqq 2\sqrt{x}$, whence

$$\sum_{q^k \| n,\, q<P(n)} k q^{\varrho} L(q) \ll x^{\frac{1}{2}(\varrho+\varepsilon)}\, \Omega(n) \ll x^{\varrho/2+\varepsilon}.$$

But if $x < n \leqq x + h$, $P(n) > \sqrt{x}$, then $n = pm$, $p = P(n) > \sqrt{x}$, $\frac{x}{m} < p \leqq \frac{x+h}{m}$, and m is an integer satisfying $1 \leqq m \leqq (x + h)/p \leqq (x + h)\, x^{-1/2} \leqq 2\sqrt{x}$, $P(m) < p$. Hence

$$\begin{aligned}\sum_{x<n\leqq x+h,\, P(n)>\sqrt{x}} P^{\varrho}(n)\, L(P(n)) &= \sum_{m\leqq (x+h)x^{-1/2}} \sum_{\frac{x}{m}<p\leqq\frac{x+h}{m},\, p>P(m)} p^{\varrho} L(p) \\ &= \sum_{m\leqq M} \sum_{\frac{x}{m}<p\leqq\frac{x+h}{m}} p^{\varrho} L(p) + O\Bigg(\sum_{M<m\leqq 2\sqrt{x}} x^{\varrho} m^{-\varrho} L(2\sqrt{x}) \sum_{\frac{x}{m}<p\leqq\frac{x+h}{m}} 1\Bigg),\end{aligned} \tag{2.1}$$

where M is a fixed, large integer. This procedure is necessary, since in the inner sum in the O-term in (2.1), $\frac{h}{m}$ is small when compared to $\frac{x}{m}$, hence we cannot use (1.7) in the range (1.8). Instead we use the well-known inequality (e.g. see H. L. Montgomery [6])

$$\pi(X + Y) - \pi(X) \leqq \frac{2Y}{\log Y} \qquad (X, Y \geqq 2) \tag{2.2}$$

to obtain

$$\sum_{\frac{x}{m}<p\leqq\frac{x+h}{m}} 1 \ll \frac{h}{m \log x} \qquad (1 \leqq m \leqq 2\sqrt{x},\ x^{7/12} \leqq h \leqq x).$$

Thus the error term in (2.1) is

$$O\left(h x^{\varrho} M^{-\varrho} \frac{L(x)}{\log x}\right). \tag{2.3}$$

We will now use Lemma 1 stated in [2] and which essentially says that there exists a real function ϕ satisfying $\lim_{x\to\infty} \phi(x) = +\infty$ and such that

$$L(x/n) = (1 + o(1))\, L(x), \qquad \text{uniformly for } 1 \leqq n \leqq \phi(x) \quad \text{as} \quad x \to \infty. \tag{2.4}$$

Hence from now on we assume that $M \leqq \phi(x)$.

We shall use now (1.7) for the range $h \geqq x^{7/12} \log^{22-\delta} x$ with x and h replaced by $\frac{x}{m}$ and $\frac{h}{m}$ respectively, $1 \leqq m \leqq M$. We have $\left(\frac{x}{m}\right)^{7/12} \log^{22-\delta} \frac{x}{m} \leqq \frac{h}{m}$ for $x^{7/12} \log^{22} x \leqq h \leqq o(x)$ as $x \to \infty$. Hence in the last range

$$\sum_{\frac{x}{m} < p \leqq \frac{x+h}{m}} p^{\varrho} L(p) \leqq \left(\frac{x+h}{m}\right)^{\varrho} L\left(\frac{x+h}{m}\right)\left(\pi\left(\frac{x+h}{m}\right) - \pi\left(\frac{x}{m}\right)\right)$$
$$= (1 + o(1)) \frac{x^{\varrho} h L(x)}{m^{\varrho+1} \log x},$$

and a lower bound of the same form follows analogously. Thus we obtain

$$\text{(2.5)} \qquad \sum_{\frac{x}{m} < p \leqq \frac{x+h}{m}} p^{\varrho} L(p) = (1 + o(1)) \frac{x^{\varrho} h L(x)}{m^{\varrho+1} \log x}$$
$$(1 \leqq m \leqq M,\ x^{7/12} \log^{22} x \leqq h \leqq o(x),\ x \to \infty).$$

Using (2.5) in (2.1) and keeping in mind (2.3) we obtain

$$\sum_{x < n \leqq x+h,\, P(n) > \sqrt{x}} P^{\varrho}(n)\, L(P(n)) = (1 + o(1)) \sum_{m \leqq M} \frac{h x^{\varrho} L(x)}{m^{\varrho+1} \log x} + O\left(h x^{\varrho} M^{-\varrho} \frac{L(x)}{\log x}\right)$$
$$= (1 + o(1))\, \zeta(\varrho + 1)\, h x^{\varrho} \frac{L(x)}{\log x} + O\left(h x^{\varrho} M^{-\varrho} \frac{L(x)}{\log x}\right),$$

and thus

$$\sum_{x < n \leqq x+h} F(n) = (\zeta(\varrho + 1) + o(1) + O(M^{-\varrho}))\, h x^{\varrho} \frac{L(x)}{\log x} + O(h x^{\varrho/2+\varepsilon})$$
$$= (\zeta(\varrho + 1) + o(1))\, h x^{\varrho} \frac{L(x)}{\log x}$$

if $0 < \varepsilon < \varrho/3$ and $M = \phi(x) \to \infty$. This finishes the proof of Theorem 1, with the remark that the $o(1)$-terms in (1.9) and (1.10) could be improved if one used a sharper form of (1.7).

3. Proof of Theorem 2. We shall prove only (1.11), since the proof of (1.12) is analogous. Using the method of proof of Theorem 1 we obtain

$$\sum_{x < n \leqq x+h} P_*(n) = \sum_{x < n \leqq x+h,\, P(n) > \sqrt{x}} P_*(n) + O(h x^{1/2+\varepsilon})$$
$$= \sum_{x < n \leqq x+h,\, P(n) > \sqrt{x}} \frac{\beta(n)}{\omega(n)} + O(h x^{1/2+\varepsilon})$$
$$= \sum_{x < n \leqq x+h,\, P(n) > \sqrt{x}} \frac{P(n)}{\omega(n)} + O(h x^{1/2+\varepsilon}).$$

Now if $x < n \leqq x + h$, $P(n) > \sqrt{x}$, then $n = pm$ with $p = P(n)$, $\omega(n) = \omega(m) + 1$, $\frac{x}{m} < p \leqq \frac{x+h}{m}$, $1 \leqq m \leqq (x + h)/p \leqq 2\sqrt{x}$, $P(m) < p$. Similarly as in the proof of (2.1), with the error term given by (2.3), we have

$$(3.1)\qquad \sum_{x<n\leqq x+h,\,P(n)>\sqrt{x}} \frac{P(n)}{\omega(n)} = \sum_{m\leqq(x+h)x^{-1/2}} \frac{1}{\omega(m)+1} \sum_{\frac{x}{m}<p\leqq\frac{x+h}{m},\,p>P(m)} p$$
$$= \sum_{1\leqq m\leqq M} \frac{1}{\omega(m)+1} \sum_{\frac{x}{m}<p\leqq\frac{x+h}{m}} p + O\left(\frac{hx}{M\log x}\right),$$

where again M is a large, fixed integer $M \leqq \phi(x)$. We use (2.5) with $\varrho = 1$, $L(x) \equiv 1$ to estimate the last sum in (3.1), and we obtain

$$(3.2)\qquad \sum_{n\leqq x+h,\,P(n)>\sqrt{x}} \frac{P(n)}{\omega(n)} = (1+o(1))\frac{hx}{\log x} \sum_{1\leqq m\leqq M} \frac{1}{m^2(\omega(m)+1)} + O\left(\frac{hx}{M\log x}\right)$$
$$= \left(\sum_{m=1}^{\infty} \frac{1}{m^2(\omega(m)+1)} + o(1)\right)\frac{hx}{\log x} + O\left(\frac{hx}{M\log x}\right).$$

Letting $M = \phi(x)$ and $x \to \infty$ we obtain (1.11) from (3.2), since analyzing the proof of (1.1) given in [1] it is found that

$$e_1 = \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{m^2(\omega(m)+1)} = 0.65321\ldots.$$

The asymptotic relation (1.12) is established analogously, with

$$f_1 = \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{m^2(\Omega(m)+1)} = 0.64259\ldots,$$

whence $0 < f_1 < e_1$. We thank J.-P. Massias (Limoges) who has kindly computed the values of e_1 and f_1.

4. Remarks. We first note that Theorem 2 could have been stated to deal with more general arithmetic functions. In fact, we can prove for instance that, using the same technique as in the proof of Theorem 2, if f is defined by (1.4) with $\varrho > 0$ and L satisfying (1.5), then, for $x^{7/12}\log^{22} x \leqq h \leqq o(x)$,

$$\sum_{x<n\leqq x+h} \frac{f(n)}{\omega(n)} = (c_\varrho + o(1))\frac{hx^\varrho L(x)}{\log x},$$

with $c_\varrho = \sum_{d=1}^{\infty} \frac{1}{(\omega(d)+1)\,d^{\varrho+1}}$.

Throughout this paper, we have focused our attention on additive functions defined by (1.4) and (1.6) where $\varrho > 0$. But one may also consider functions with corresponding $\varrho = 0$, which means dealing with "small" additive functions, such as $\omega(n)$ and $\Omega(n)$ (whereas for $\varrho > 0$, we had "large" additive functions). In the case of $\sum_{2\leqq n\leqq x} f(n)$, this possibility was treated in detail in [2]. There, essentially, it was shown that if $f(n) = \sum_{p\mid n} L(p)$, where L is a slowly oscillating function of the type (1.5), then

$\sum_{2\leqq n\leqq x} f(n) \sim x \int_2^x \frac{L(u)}{u \log u}\,du$, and that the asymptotic behaviour of $U(x) := \int_2^x \frac{L(u)}{u \log u}\,du$ can be given more explicitly depending on the value of $c = \lim_{x\to\infty} x\,(\log x)\,L'(x)/L(x)$. In fact, five cases for the values of c were considered, namely $c = \infty$, $c > 0$, $c = 0$, $c < 0$ and $c = -\infty$. In three of these, $U(x)$ can be simplified. For example, if $c > 0$, one has $U(x) \sim c^{-1} L(x)$, while if $c < 0$ or $c = -\infty$, one has $U(x) \sim \sum_p f(p)/p$.

Appropriate analogues of (1.9) and (1.10) may also be obtained in the case $\varrho = 0$, provided the corresponding function $L(x)$ grows fast enough, that is, if it satisfies

$$\lambda(x) = \lambda_L(x) = x\,(\log x)\,L'(x)/L(x) \to +\infty, \quad \text{as } x \to \infty\,. \tag{4.1}$$

In order to avoid repetition of some of the arguments already given above, we only state the result and give the main idea of the proof.

Theorem 3. *Let $f(n)$ and $F(n)$ be defined by* (1.4) *and* (1.6) *with $\varrho = 0$ respectively, with $L(x)$ given by* (1.5) *and satisfying* (4.1). *Then, for $x^{7/12+\varepsilon} \leqq h \leqq o(x)$, we have, as $x \to \infty$,*

$$\sum_{x<n\leqq x+h} f(n) = (1 + o(1))\,h \int_2^x \frac{L(u)}{u \log u}\,du\,, \tag{4.2}$$

and the same estimate holds for $\sum_{x<n\leqq x+h} F(n)$.

We only sketch the proof of (4.2). First we write

$$\sum_{x<n\leqq x+h} f(n) = \sum_{p\leqq x+h} L(p)\left(\left[\frac{x+h}{p}\right] - \left[\frac{x}{p}\right]\right) = \sum_{m\leqq x}\ \sum_{\frac{x}{m}<p\leqq\frac{x+h}{m}} L(p)$$
$$= \sum\nolimits_1 + \sum\nolimits_2,$$

where in $\sum_1$, we have $m \leqq x^\varepsilon$ and in $\sum_2$, $x^\varepsilon < m \leqq x$. In $\sum_1$ the condition $h \geqq x^{7/12+\varepsilon}$ ensures that

$$\frac{h}{m} \geqq \left(\frac{x}{m}\right)^{7/12} \left(\log \frac{x}{m}\right)^{22}.$$

Hence we may use (1.7) (with x and h replaced by x/m and h/m, respectively) to deduce that the inner sum in $\sum_1$ is equal to

$$(1 + o(1))\,\frac{L(x/m)\cdot h/m}{\log(x/m)}\,.$$

This allows us to write

$$\sum\nolimits_1 = (1 + o(1))\,h \int_2^{x^\varepsilon} \frac{L(x/t)\,dt}{t \log(x/t)} = (1 + o(1))\,h(U(x) - U(x^{1-\varepsilon}))\,,$$

where we put

$$U(x) = U_L(x) = \int_2^x \frac{L(u)}{u \log u}\,du\,.$$

Clearly $U(x) \to \infty$ as $x \to \infty$, and by l'Hôpital's rule we obtain ($0 < \varepsilon < 1$ is fixed)

$$\lim_{x \to \infty} \frac{U(x)}{U(x^{1-\varepsilon})} = \lim_{x \to \infty} \frac{U'(x)}{(1-\varepsilon)\, x^{-\varepsilon} U'(x^{1-\varepsilon})} = \lim_{x \to \infty} \frac{L(x)}{L(x^{1-\varepsilon})}$$

$$= \lim_{x \to \infty} \exp\left(\int_{x^{1-\varepsilon}}^{x} \eta(t)\, t^{-1}\, dt \right) = \exp\left(\lim_{x \to \infty} \int_{x^{1-\varepsilon}}^{x} \frac{\lambda(t)}{t \log t}\, dt \right)$$

$$\geqq \exp\left(\lim_{x \to \infty} \left(\min_{x^{1-\varepsilon} \leqq t \leqq x} \lambda(t) \right) \int_{x^{1-\varepsilon}}^{x} \frac{dt}{t \log t} \right)$$

$$\geqq \exp\left(\lim_{x \to \infty} \left(\min_{x^{1-\varepsilon} \leqq t \leqq x} \lambda(t) \right) \cdot \log \frac{1}{1-\varepsilon} \right) = \infty,$$

since $\lambda(t) \to \infty$ as $t \to \infty$, and $\eta(t) = \lambda(t)/\log t$ by (1.5) and (4.1). Hence $U(x^{1-\varepsilon}) = o(U(x))$ as $x \to \infty$, and this proves that

$$\sum\nolimits_1 = (1 + o(1))\, h\, U(x) \qquad (x \to \infty).$$

And finally, using (2.2), we obtain that

$$\sum\nolimits_2 < \sum_{x^\varepsilon < m \leqq x} L(x/m) \left(\pi\left(\frac{x+h}{m}\right) - \pi\left(\frac{x}{m}\right) \right) < 2h \sum_{x^\varepsilon < m \leqq x} \frac{L(x/m)}{m \log(h/m)}$$

$$\ll h \sum_{x^\varepsilon < m \leqq x} \frac{L(x/m)}{m \log(x/m)} \ll h\, U(x^{1-\varepsilon}) = o(h\, U(x)),$$

so that (4.2) is established.

It is also interesting to note that, using a recent result of C. J. Mozzochi [7], namely

$$\pi(x+h) - \pi(x) \gg h/(\log x), \qquad x^{c+\varepsilon} \leqq h \leqq x,$$

with $c = \frac{11}{20} - \frac{1}{384} = 0.5473958\ldots$, one can replace the estimates (1.9), (1.10), (1.11), (1.12) and (4.2) by lower bounds for the corresponding sums for a wider range of $h = h(x)$.

References

[1] J.-M. De Koninck and A. Ivić, The distribution of the average prime divisor of an integer. Arch. Math. **43**, 37–43 (1984).

[2] J.-M. De Koninck et A. Mercier, Les fonctions arithmétiques et le plus grand facteur premier. Acta Arith., in print.

[3] A. Ivić, On sums of large differences between consecutive primes. Math. Ann. **241**, 1–9 (1979).

[4] A. Ivić, The Riemann zeta-function. New York 1985.

[5] J. Karamata, Sur un mode de croissance régulière des fonctions. Mathematica (Cluj) **4**, 38–53 (1930).

[6] H. L. Montgomery, Topics in multiplicative number theory. LNM **227**, Berlin-Heidelberg-New York 1971.

[7] C. J. Mozzochi, On the difference between consecutive primes. J. Number Theory **24**, 181–187 (1986).
[8] E. Seneta, Regularly varying functions. LNM **508**, Berlin-Heidelberg-New York 1976.

Eingegangen am 16. 6. 1987 *)

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