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SUMS INVOLVING THE LARGEST PRIME DIVISOR OF AN INTEGER II

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For $n \geqslant 2$, let $P(n)$ denote the largest prime divisor of an integer. In this paper, we develop an elementary method for estimating the sum $\sum_{2 \leqslant n \leqslant x} f(n) P(n)$ where $f(n)$ is a multiplicative arithmetical function.

1. INTRODUCTION

Let $P(n)$ denote the largest prime divisor of an integer $n \geqslant 2$. Various sums involving $P(n)$ have been studied by many, authors[1-5,7,8,10-12,15]. In this paper, we discuss sums of the form $\sum_{2 < n \leqslant x} f(n) P(n)$ where $f(n)$ is a multiplicative function. Our methods are elementary and can also be used to estimate sums of the form $\sum_{2 \leqslant n \leqslant x} f(n) g(n)$ where $f(n)$ is multiplicative and $g(n)$ is additive. Through analytic methods are available for estimating the latter sums[9,4,13,14], it appears that elementary methods are best suited for the sums discussed in this paper.

2. PRELIMINARIES

Let $\mu(n)$ denote the Möbius function $\phi(n)$ be the Euler totient function, $\sigma(n)$ be the sum of all the positive divisors of n, $\omega(n)$ be the number of distinct prime factors of n and for $k \geqslant 1$, $d_k(n)$ be the Piltz divisor function defined to be the number of ordered k-tuples $(x_1, x_2, ..., x_k)$ of positive integers such that $x_1, x_2, x_k = n$. Also let $\beta(n) = \sum_{p \mid n} p$ and $B(n) = \sum_{p^\alpha \| n} \alpha p$ where, as usual, P denotes a prime number and $p^\alpha \| n$ means that $p^\alpha | n$ and $p^{\alpha+1} \nmid n$. Further for any function g defined at primes, let the function G be defined by $G(I) = 0$ and for $n > 1$

$$G(n) = \sum_{p \mid n} g(p). \qquad \ldots(2.1)$$

In the sequel, $\sum'_{n \leqslant x}$ means $\sum_{2 < n \leqslant x}$.

Lemma 2.1—As $x \to \infty$, we have

$$\sum_{n \leqslant x}{}' P(n) = \sum_{n \leqslant x} \beta(n) + O\left[\frac{x^{3/2}}{\log x}\right] \qquad \ldots(2.2)$$

and

$$\sum_{n \leqslant x} B(n) = \sum_{n < x} \beta(n) + O(x \log \log x). \qquad \ldots(2.3)$$

PROOF : We have

$$\sum_{n \leqslant x} P(n) = \sum_{pm \leqslant x, p(m) < p} p = \sum_{\substack{pm \leqslant x \\ p(m) < p, p \leqslant \sqrt{x}}} P + \sum_{\substack{pm < x \\ p(m) \leqslant p, p) > \sqrt{x}}} p$$

$$= O\left[\sum_{pm \leqslant x, p \leqslant \sqrt{x}} p\right] + \sum_{pm < x} p - \sum_{\substack{pm < x \\ p < \sqrt{x}}} p$$

$$= \sum_{n \leqslant x} \beta(n) + O\left[\sum_{p < \sqrt{x}} p \frac{x}{p}\right]$$

and (2.2) follows. On noting that

$$\sum_{n \leqslant x} B(n) = \sum_{n \leqslant x} \beta(n) + \sum_{p^{\alpha} m \leqslant x, \alpha \geqslant 2} \alpha p$$

$$= \sum_{n < x} \beta(n) + O\left[x \sum_{p^{\alpha} \leqslant x, \alpha \geqslant 2} \alpha / p^{\alpha - 1}\right]$$

we get (2.3).

Theorem 2.1—Let $f(n)$ be an arithmetical function such that $f(n) \leqslant g(n)$ where $g(x)$ is positive valued and increasing on $[1, \infty]$. Then

$$\sum_{n < x} f(n) \beta(n) = \sum_{n < x}{}' f(n) P(n) + O(g(x) x^{3/2} / \log x). \qquad \ldots(2.4)$$

$$\sum_{n < x} f(n) B(n) = \sum_{n \leqslant x} f(n) \beta(n) + O(g(x) x \log \log x) \qquad \ldots(2.5)$$

$$\sum_{n \leqslant x} f(n) B(n) = \sum_{n \leqslant x}{}' f(n) P(n) + O(g(x) x^{3/2} / \log x). \qquad \ldots(2.6)$$

PROOF : Since

$$\sum_{n \leqslant x} f(n) \beta(n) = \sum_{n \leqslant x}{}' f(n) P(n) + \sum_{n \leqslant x}{}' f(n) (\beta(n) - P(n)) + O(1)$$

$$= \sum_{n < x}{}' f(n) P(n) + O\left(g(x) \sum_{n \leqslant x}{}' (\beta(n) - P(n))\right) + O(1)$$

(2.4) follows from (2.2). Similarly (2.5) follows from (2.3) and (2.6) follows from (2.4) and (2.5).

Remark 2.1 : Theorem 2.1 extends and sharpens an earlier result due to (De Koninck and Ivić[5], Lemma 3).

Lemma 2.2—Let $G(n)$ be as defined in (2.1). Then

$$\sum_{n\leqslant x} d_k(n)\, G(n) = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \Big[\sum_{p^i n\leqslant x} g(P)\, d_k(n)\Big].$$

PROOF : We recall (cf. Ivic and Pomerance[15], p. 4-5) that $d_k(n)$ is multiplicative and $\hat{d}_k(p^\alpha) = \left[\begin{matrix}\alpha+k-1\\ \alpha\end{matrix}\right]$ for prime p and positive integral α. Also we note the following identity :

$$\sum_{i,j\geqslant 0,\, i+j=n} (-1)\, i \left[\begin{matrix}k\\ i\end{matrix}\right] \left[\begin{matrix}k+j-1\\ j\end{matrix}\right] = 0 \text{ for } k\geqslant 1 \text{ and } n\geqslant 1. \quad ...(2.7)$$

This follows from

$$1 = (1-z)^k (1-z)^{-k} = \left\{\sum_{i=0}^{k} (-1)^i \left[\begin{matrix}k\\ i\end{matrix}\right] z^i\right\} \left\{\sum_{j=0}^{k} \left[\begin{matrix}k+j-1\\ j\end{matrix}\right] z^j\right\}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{i,j\geqslant 0,\, i+j=n} (-1)^i \left[\begin{matrix}k\\ i\end{matrix}\right]\left[\begin{matrix}k+j-1\\ j\end{matrix}\right]\right\} z^n.$$

Now by (2.1)

$$\sum_{n\leqslant x} d_k(n)\, G(n) = \sum_{n\leqslant x} d_k(n) \sum_{p|n} g(p) = \sum_{pn\leqslant x} g(p)\, d_k(pn)$$

$$= \sum_{pn\leqslant x,\, p\nmid n} g(p)\, d_k(n) + \sum_{pn\leqslant x,\, p\mid n} g(p)\, d_k(pn)$$

$$= \left[\begin{matrix}k\\ i\end{matrix}\right] \left\{ \sum_{pn\leqslant x} g(p)\, d_k(n) - \sum_{pn\leqslant x,\, p\mid n} g(p)\, d_k(n) \right\}$$

$$+ \sum_{pn\leqslant x,\, p\mid n} g(p)\, d_k(pn)$$

$$= \left[\begin{matrix}k\\ 1\end{matrix}\right] \sum_{pn\leqslant x} g(p)\, d_k(n) + \sum_{p^2 n\leqslant x} g(p) \left\{ d_k(P^2 n) \right.$$

$$\left. - \left[\begin{matrix}k\\ 1\end{matrix}\right] d_k(pn) \right\}$$

$$= \left[\begin{matrix}k\\ 1\end{matrix}\right] \sum_{pn\leqslant x} g(p)\, d_k(n) + \Sigma_1 \quad ...(2.8)$$

say, Now since $\begin{bmatrix} k+1 \\ 2 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix}^2 = - \begin{bmatrix} k \\ 2 \end{bmatrix}$, we have

$$\sum\nolimits_1 = - \begin{bmatrix} k \\ 2 \end{bmatrix} \sum_{p^2 n < x, p \nmid n} g(p)\, d_k(n) + \sum_{p^3 n < x} g(p) \Big\{ d_k(p^3 n)$$

$$= - \begin{bmatrix} k \\ 1 \end{bmatrix} d_k(p^n\, n) = - \left\{ \begin{matrix} k \\ 2 \end{matrix} \right\} \sum_{p^2 n < x} g(p)\, d_k(n) + \sum_{p^3 n < x} g(p)$$

$$\Big\{ d_k(p^3 n) - \begin{bmatrix} k \\ 1 \end{bmatrix} d_k(p^2 n) + \begin{bmatrix} k \\ 2 \end{bmatrix} d_k(pn) \Big\}.$$

Continuing this, we arrive, by (2.8), at

$$\sum_{n < x} d_k(n)\, G(n) = \sum_{i=1}^{k} (-i)^{i-1} \begin{bmatrix} k \\ i \end{bmatrix} \sum_{p^i n < x} g(p)\, d_k(n)$$

$$+ \sum_{p^{k+1} n \leqslant x} g(p) \Big\{ d_k(p^{k+1}) - \begin{bmatrix} k \\ 1 \end{bmatrix} d_k(p^k n) + \ldots + (-1)^k \begin{bmatrix} k \\ k \end{bmatrix} d_k(pn) \Big\}$$

However, this last sum is zero since for any prime p and any integer $\alpha \geqslant 0$

$$d_k(p^{k+1+\alpha}) - \begin{bmatrix} k \\ 1 \end{bmatrix} d_k(p^{k+\alpha}) + \ldots + (-1)^k \begin{bmatrix} k \\ k \end{bmatrix} d_k(p^{1+\alpha}) = 0.$$

In fact, this is equivalent to

$$\begin{bmatrix} \overline{k+1+\alpha}+k-1 \\ k+1+\alpha \end{bmatrix} - \binom{k}{1} \begin{bmatrix} \overline{k+\alpha}+k-1 \\ k+\alpha \end{bmatrix} + \cdots + (-1)\, k - \begin{bmatrix} k \\ k \end{bmatrix}$$

$$\begin{bmatrix} \overline{1+\alpha}+k-1 \\ 1+\alpha \end{bmatrix} = 0$$

which is the case $n = k+1+\alpha$ of the identity (2.7). This completes the proof of the Lemma.

Lemma 2.3—Let G be defined as in (2.1). Then

$$\sum_{n < x} \mu(n)\, G(n) = - \sum_{p^\alpha n \leqslant x} g(p)\, \mu(n) \qquad \ldots(2.9)$$

$$\sum_{n > x} \phi(n)\, G(n) = \sum_{p^\alpha n \leqslant x} (p-1)\, g(p)\, \phi(n) \qquad \ldots(2.10)$$

and if F is any multiplicative function satisfying

$$F(p^{\alpha+2}) + pF(p^\alpha) = F(p)\, F(p^{\alpha+1}) \qquad \ldots(2.11)$$

for all primes and positive integers α, then

$$\sum_{n\leqslant x} F(n)\, G(n) = \sum_{pn\leqslant x} g(p)\, F(p)\, F(n) - \sum_{p^2 n\leqslant x} pg(p)\, F(n). \quad \ldots(2.12)$$

PROOF: We prove (2.9), the proofs of (2.10) and (2.12) being similar.

$$\sum_{n\leqslant x} \mu(n)\, G(n) = \sum_{n\leqslant x} \mu(n) \sum_{p\mid n} g(p) = \sum_{pn\leqslant x} g(P)\, \mu(pn)$$

$$= - \sum_{pn\leqslant x, p\nmid n} g(p)\, \mu(n) = - \sum_{pn\leqslant x} g(p)\, \mu(n)$$

$$+ \sum_{pn\leqslant x, p\mid n} g(p)\, \mu(n)$$

$$= - \sum_{pn\leqslant x} g(p)\, \mu(n) + \sum_{p^2 n\leqslant x} g(p)\, \mu(pn)$$

$$= - \sum_{pn\leqslant x} g(p)\, \mu(n) - \sum_{p^2 n\leqslant x, p\nmid n} g(p)\, \mu(n)$$

$$= - \sum_{pn\leqslant x} g(p)\, \mu(n) - \sum_{p^2 n\leqslant x} g(p)\, \mu(n)$$

$$+ \sum_{p^2 n\leqslant x} g(p)\, \mu(n).$$

Continuing this, we get (2.9).

3. MAIN RESULTS

In the following, N denotes an arbitrary but fixed positive integer.

Theorem 3.1—There exist constants $a_0 = \frac{\pi^2}{12}$, $a_1, \ldots, a_{N-1}$ such that

$$\sum_{n\leqslant x}{}' P(n) = x^2 \sum_{i=1}^{N} \frac{a_{i-1}}{(\log x)^i} + O(x^2/\log x)^{N+1}) \quad \ldots(3.1)$$

$$\sum_{n\leqslant x} \beta(n) = x^2 \sum_{i=1}^{N} \frac{a_{i-1}}{(\log x)^i} + O(x^2/(\log x)^{N+1}) \quad \ldots(3.2)$$

and

$$\sum_{n\leqslant x} B(n) = x^2 \sum_{i=1}^{N} \frac{a_{i-1}}{(\log x)^i} + O(x^2/(\log x)^{N+1}). \quad \ldots(3.3)$$

Further, we have

$$a_i = \int_1^\infty \frac{[t]\,(\log t)^i}{t^3}\, dt \quad \ldots(3.4)$$

where $[x]$ denotes the largest integer $\leqslant x$ and

$$a_0 < a_1 < a_2 < \dots < a_{N-1}. \qquad \dots(3.5)$$

PROOF : We prove (3.2) and note that Lemma 2.1 and (3.2) yield (3.1) and (3.3). By the prime number theorem and partial summation, we have

$$\sum_{p \leqslant x} p = \int_2^x \frac{t}{\log t} dt + O\left[\frac{x^2}{(\log x)^{N+1}}\right]. \qquad \dots(3.6)$$

Hence by partial summation

$$\sum_{n \leqslant x} \beta(n) = \sum_{pn \leqslant x} p + \sum_{n \leqslant x/2} \left[\sum_{p < x/n} p\right]$$

$$= \sum_{n \leqslant x/2} \left\{ \int_2^{x/n} \frac{t}{\log t} \, dt + O\left[\frac{x^2}{n^2 (\log x/n)^{N+1}}\right]\right\}$$

$$= x^2 \int_1^{x/2} \frac{[t]}{t^3 \log(x/t)} \, dt + O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

$$= \frac{x^2}{\log x} \int_1^{x/2} \frac{[t]}{t^3} \left\{ \sum_{i=0}^{N-1} \left[\frac{\log t}{\log x}\right]^i + O\left[\left[\frac{\log t}{\log i}\right]^N\right]\right\} dt + O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

$$= \frac{x^2}{\log x} \sum_{i=0}^{N-1} \frac{1}{(\log x)^i} \left\{ \int_1^{\infty} \frac{[t] (\log t)^i}{t^3} \, dt + O\left[\frac{(\log x)^i}{x}\right]\right\}$$

$$+ O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

$$= x^2 \sum_{i=1}^{N} \frac{a_i - 1}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

where a_i is as given in (3.4).

To prove (3.5) we have by partial summation

$$\sum_{n \leqslant x} \frac{1}{n^2} = \frac{[x]}{x^2} + 2 \int_1^x \frac{[t]}{t^3} \, dt.$$

So that on letting $x \to \infty$, we get $\zeta(2) = 2a_0$. By a similar argument, we find

$$\sum_{n=1}^{\infty} \frac{(\log n)^i}{n^2} = 2a_i - ia_{i-1}, \; i > 1. \qquad \dots(3.7)$$

It is easy to show that $\zeta(2) < 2\sum_{n=1}^{\infty}\frac{\log n}{n^2}$ so that $a_0 < a_1$. For $i \geqslant 2$, we have by (3.7)

$$2a_i = ia_{i-1} + \sum_{n=2}^{\infty}\frac{(\log n)^i}{n^2} > ia_{i-1} \geqslant 2a_{i-1}$$

since $a_i \geqslant 0$ for all i. This proves (3.5) and thus the theorem.

Remark 3.1: Theorem 3 1 is due to (De Koninck and Ivić[5], Theorem 1) who sharpened an earlier result of Alladi and Erdös[2]. De Koninck and Ivić proved (3.1) and deduced (3.2) and (3.3). Our method of proof has an advantage in that it yields the representation (3.4) (and the inequalities given in (3.5)).

Theorem 3.2—There exist constants $d_0, d_1, \ldots, d_{N-1}$ such that

$$\sum_{n\leqslant x}{}' d_k(n)\, P(n) = x^2\sum_{i=1}^{N}\frac{d_{i-1}}{(\log x)^i} + O\frac{x^2}{(\log x)^{N+1}}\Big] \qquad \ldots(3.8)$$

$$\sum_{n\leqslant x} d_k(n)\, B(n) = x^2\sum_{i=1}^{N}\frac{d_{-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.9)$$

$$\sum_{n\leqslant x} d_k(n)\, B(n) = x^2\sum_{i=1}^{N}\frac{d_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.10)$$

PROOF : First we prove (3.9). By Lemma 2.2, we have

$$\sum_{n\leqslant x} d_k(n)\, \beta(n) = \sum_{i=1}^{k}(-1)^{i-1}\left[\begin{matrix}k\\ i\end{matrix}\right]\sum_{p^i n\leqslant x} p d_k(n)$$

$$= \left[\begin{matrix}k\\ 1\end{matrix}\right]\sum_{pn\leqslant x} p d_k(n) + \sum_{i=2}^{k}(-1)^{i-1}\left[\begin{matrix}k\\ i\end{matrix}\right] \times \sum_{p^i n\leqslant x} p d_k(n)$$

$$= k\,\Sigma_1 + \sum_{i=2}^{k}(-1)^{i-1}\left[\begin{matrix}k\\ i\end{matrix}\right]\Sigma_i \qquad \ldots(3.11)$$

say. Since $\sum_{n \leqslant x} d_k(n) \ll_k x(\log x)^{k-1}$ (cf. Ivić and Pomerance[15], p. 263), we have for $i \geqslant 2$

$$\Sigma_i = \sum_{p^i n \leqslant x} p\, d_k(n) = \sum_{p^i \leqslant x} p \sum_{n \leqslant x/p^i} d_k(n)$$

$$\ll \sum_{p^i \leqslant x} p\left(\frac{x}{p^i}\right)\left(\log \frac{x}{p^i}\right)^{k-1} \ll x(\log x)^{k-1} \log\log x. \tag{3.12}$$

For the estimation of Σ_1, we note that by (3.6), for each positive integer r, there exist constants $b_1, b_2, \ldots, b_r$ such that

$$\sum_{p \leqslant x} p = x^2 \sum_{i=1}^{r} \frac{b_i}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{r+1}}\right]. \quad \ldots(3.13)$$

Hence

$$\Sigma_1 = \sum_{pn \leqslant x, n \leqslant \sqrt{x}} pd_k(n) + \sum_{pn \leqslant x, n > \sqrt{x}} pd_k(n)$$

$$= \sum_{n \leqslant \sqrt{x}} d_k(n) \sum_{p \leqslant x/n} p + O\left[\sum_{n > \sqrt{x}} d_k(n) \frac{x^2}{n^2}\right]$$

$$= \sum_{n \leqslant \sqrt{x}} d_k(n) \left\{\frac{x^2}{n^2} \sum_{i=1}^{r} b_i \left[\log \frac{x}{n}\right]^{-i} + O\left[\frac{(x/n)^2}{(\log x/n)^{r+1}}\right]\right\}$$

$$+ O(x^{3/2}(\log x)^{k-1})$$

$$= x^2 \sum_{i=1}^{r} \frac{b_i}{(\log x)^i} \left\{\sum_{n \leqslant \sqrt{x}} \frac{d_k(n)}{n^2}\left[1 - \frac{\log n}{\log x}\right]^{-i}\right\} + O\left[\frac{x^2}{(\log x)^{r+1}}\right]$$

$$= x^2 \sum_{i=1}^{r} \frac{b_i}{(\log x)^i} \left\{\sum_{n \leqslant \sqrt{x}} \frac{d_k(n)}{n^2}\left[\sum_{j=0}^{r} b_{ij} \left[\frac{\log n}{\log x}\right]^j\right.\right.$$

$$\left.\left. + O\left[\left[\frac{\log n}{\log x}\right]^{r+1}\right]\right]\right\} + O\left[\frac{x^2}{(\log x)^{r+1}}\right]$$

$$= x^2 \sum_{\substack{1 \leqslant i \leqslant r \\ 0 \leqslant j \leqslant r}} \frac{b_i b_{ij}}{(\log x)^{i+j}} \left\{\sum_{n=1}^{\infty} \frac{d_k(n)(\log n)^j}{n^2} + O\left[\frac{(\log x)^{k-1}}{x^{1/2}}\right]\right\}$$

$$+ O\left[\frac{x^2}{(\log x)^{r+1}}\right]$$

$$= x^2 \sum_{i=1}^{r} \frac{c_i}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{r+1}}\right].$$

Now (3.9) follows from (3.11), (3.12) and the above.

On taking $f(n) = d_k(n)$, $g(x) = x^\epsilon$ and recalling that $d_k(n) \ll_\epsilon n^\epsilon$ for each $\epsilon > 0$) in Theorem 2.1, we obtain (3.8) and (3.10) from (3.9).

Theorem 3.3—There exist constants e_i, f_i, g_i, $0 \leqslant i \leqslant N-1$ such that

$$\sum_{n \leqslant x}{}' \mu(n) P(n) = x^2 \sum_{i=1}^{N} \frac{e_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.14)$$

$$\sum_{n \leqslant x} \mu(n) \beta(n) = x^2 \sum_{i=1}^{N} \frac{e_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.15)$$

$$\sum_{n \leqslant x} \mu(n) B(n) = x^2 \sum_{i=1}^{N} \frac{e_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.16)$$

$$\sum_{n < x}{}' \phi(n) P(n) = x^3 \sum_{i=1}^{N} \frac{f_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.17)$$

$$\sum_{n \leqslant x} \phi(n) \beta(n) = x^3 \sum_{i=1}^{N} \frac{f_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.18)$$

$$\sum_{n \leqslant x} \mu(n) B(n) = x^3 \sum_{i=1}^{N} \frac{f_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.19)$$

$$\sum_{n \leqslant x}{}' \sigma(n) P(n) = x^3 \sum_{i=1}^{N} \frac{g_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.20)$$

$$\sum_{n \leqslant x} \mu(n) \beta(n) = x^3 \sum_{i=1}^{N} \frac{g_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.21)$$

and

$$\sum_{n \leqslant x} \sigma(n) B(n) = x^3 \sum_{i=1}^{N} \frac{g_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right]. \qquad (3.22)$$

PROOF : The proofs of (3.15), (3.18) and (3.21) are similar to that of (3.9). We use (2.9), (2.10) and (2.12) in turn, instead of Lemma 2.2. The remaining assertions of the theorem follow from Theorem 2.1.

4. Other Results

Our general summation formulae given in Lemmas 2.2 and 2.3 and our methods of Section 2 can be used to discuss sums of the form $\sum_{n\leqslant x} f(n)\, g(n)$ where $f(n)$ is either d_k or $\mu(n)$ or $\phi(n)$ or $\sigma(n)$ and $g(n)$ is an additive function. As a further example, we have

Theorem 4.1—For any integers $k \geqslant 2$ and $N \geqslant 1$, there exist constants A_i, B_i and C_j, $0 \leqslant i \leqslant k-1$ and $0 \leqslant j \leqslant N-1$ such that

$$\sum_{n\leqslant x} d_k(n)\,\omega(n) = x \log\log x \sum_{i=0}^{k-1} A_i (\log x)^{k-1-i} + x \sum_{i=0}^{k-1} B_i (\log x)^{k-1-i}$$

$$+ x \sum_{i=0}^{N-1} \frac{C_i}{(\log x)^i} + O\left[\frac{x}{(\log x)^N}\right].$$

Proof : It is well known (cf. Ivić and Pomerance[15], p. 163) that for certain constants $a_i^{(k)}$, $0 \leqslant i \leqslant k-1$

$$\sum_{n\leqslant x} d_k(n) = x \sum_{i=0}^{k-1} a_i^{(k)} (\log x)^{k-1-i} + O[x^{1-1/k}]. \qquad \ldots(4.1)$$

Now by Lemma 2.2

$$\sum_{n\leqslant x} d_k(n)\,\omega(n) = \sum_{i=1}^{k} (-1)^{i-1} \begin{bmatrix} k \\ i \end{bmatrix} \Big\{ \sum_{p^i n\leqslant x} d_k(n) \Big]$$

$$= \begin{bmatrix} k \\ i \end{bmatrix} \Sigma_1 + \sum_{i=2}^{k} (-1)^{i-1} \begin{bmatrix} k \\ i \end{bmatrix} \Sigma_i \qquad \ldots(4.2)$$

say.

For $i \geqslant 2$, we have by (4.1)

$$\sum\nolimits_i = \sum_{p\leqslant x^{1/i}} \sum_{n\leqslant x/p^i} d_k(n) = \sum_{p\leqslant x^{1/i}} \Big\{ \frac{x}{p^i} \sum_{j=0}^{k-1} a_j^{(k)} \Big[\log \frac{x}{p^i}\Big]^{k-1-j}$$

$$+ O\Big\{\Big[\frac{x}{p^i}\Big]^{1-1/k}\Big]\Big\}$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{p\leqslant x^{1/i}} \frac{1}{p^i} \Big\{ \sum_{r=0}^{k-1} (-1)^r i^r \begin{bmatrix} k-1-j \\ r \end{bmatrix} (\log x)^{k-1-j-r}$$

$$\times (\log p)^r \Big\} + O\Big[x^{1-1/k} \sum_{p\leqslant x^{1/i}} p^{-i(1-1/k)}\Big]$$

(*equation continued on p.* 1000)

$$= x\sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=0}^{k-1-j} (-1)^{r} {}_{i}{}^{r} \left[\begin{matrix} k-1-j \\ r \end{matrix}\right] (\log x)^{k-1-j-r} \left\{ \sum_{p \leqslant x^{1/i}} \frac{\log p)^r}{p^i} \right\}$$

$$+ O\,[x^{1-1/k} \log\log x]$$

$$= x\sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=0}^{k-1-j} (-1)^{r} {}_{i}{}^{r} \left[\begin{matrix} k-1-j \\ r \end{matrix}\right] (\log x)^{k-1-j-r} \left\{ \sum_{p} \frac{(\log p)^r}{p^i} \right.$$

$$\left. + O\left[\frac{(\log x)^r}{x^{1-1/i}} \right] \right\} + O\,[x^{1-1/k} \log\log x]$$

$$= x \sum_{j=0}^{k-1} D_j\, (\log x)^{k-1-j} + O\ (x^{1-1/k} \log\log x] \qquad \ldots(4.3)$$

for certain constants D_j, $0 \leqslant j \leqslant k-1$.

For the estimation of Σ_1 we use Dirichlet's hyperbola method.

$$\Sigma_1 + \sum_{pn \leqslant x} d_k(n) = \sum_{\substack{pn \leqslant x \\ n < \sqrt{x}}} d_k(n) + \sum_{\substack{pn \leqslant x \\ n \leqslant \sqrt{x}}} d_k(n) - \sum_{pn \leqslant x} d_k(n) - \sum_{p \leqslant \sqrt{x}, n \leqslant \sqrt{x}} d_k(n)$$

$$= \sum\nolimits_1^{(1)} + \sum\nolimits_1^{(2)} - \sum\nolimits_1^{(3)}, \qquad \ldots(4.3)$$

say.

Now it is known by elementary methods that for every $C > 0$

$$\sum_{n \leqslant x} \frac{1}{P} \log\log x + B + O\left[\frac{1}{(\log x)\, c} \right] \qquad \ldots(4.5)$$

where B is a constant. Hence

$$\sum\nolimits_1^{(1)} = \sum_{pn \leqslant x, p \leqslant \sqrt{x}} d_k(n) = \sum_{p \leqslant \sqrt{x}} \sum_{n \leqslant x/p} d_k(n)$$

$$= \sum_{p \leqslant \sqrt{x}} \left\{ \frac{x}{P} \sum_{j=0}^{k-1} a_j^{(k)} \left[\log \frac{x}{P} \right]^{k-1-j} + O\left[\left[\frac{x}{P} \right]^{1-1/k} \right] \right\}$$

$$= \sum_{j=0}^{k-1} a_j^{(k)} \sum_{p \leqslant n\sqrt{x}} \frac{x}{P} \sum_{r=0}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ r \end{matrix}\right] (\log x)^{k-1-j-r} (\log P)^r$$

$$+ O\left[x^{1-1/2k} \right]$$

(*equation continued on p.* 1001)

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{j=0}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ r \end{matrix} \right] (\log x)^{k-1-j-r} \sum_{p \leq \sqrt{x}} \frac{(\log P)^r}{p}$$

$$+ O\left[x^{1-1/2k} \right]$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^{k-1-j} \left\{ \log \log x - \log 2 + B + O\left(\frac{1}{(\log x)^c} \right) \right\}$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=1}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ r \end{matrix} \right] (\log x)^{k-1-j-r} \left\{ \frac{(\log x)^r}{2^r . r} + E_r \right.$$

$$\left. + O\left[\frac{1}{(\log x)^c} \right] \right\} + O\left[x^{1-1/2k} \right] \qquad \ldots(4.6)$$

$$= x \log \log x \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^{k-1-j} + x \sum_{j=0}^{k-1} b_j^{(k)} (\log x)^{k-1-j}$$

$$+ O\left[\frac{x}{(\log p^{c+1+k}} \right].$$

For the estimations of $\sum_1^{(2)}$ and $\sum_1^{(3)}$ we write $D_k(x) = \sum_{n \leq x} d_k(n)$ and $\Delta_k(x)$ $= D_k(x) - x \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^{k-1-j}$. Then

$$\Sigma_1^{(3)} = \left[\sum_{p \leq \sqrt{x}} 1 \right] \left[\sum_{n \leq \sqrt{x}} d_k(n) \right] = \pi(\sqrt{x})\, D_k(\sqrt{x})$$

$$= \left[\text{Li}(\sqrt{x}) + O\left[\frac{\sqrt{x}}{(\log x)^c} \right] \right] D_k(\sqrt{x})$$

$$= \text{Li}(\sqrt{x})\, D_k(\sqrt{x}) + O\left[x/(\log x)^{c+1-k} \right] \qquad \ldots(4.7)$$

for any $c > 0$, by the prime number theorem. Here $\text{Li } x = \int_2^x \frac{dt}{\log t}$. Also

$$\sum_1^{(2)} = \sum_{n \leq \sqrt{x}} d_k(n) \sum_{p \leq x/n} 1 = \sum_{n \leq \sqrt{x}} d_k(n) \left\{ \text{Li} \left[\frac{x}{n} \right] + O\left[\frac{x/n}{(\log x/n)^c} \right] \right\}$$

$$= \sum_{n \leq \sqrt{x}} d_k(n)(n) \text{ Li}\left[\frac{x}{n} \right] + O\left[\frac{x}{(\log x/n^{c-k}} \right]. \qquad \ldots(4.8)$$

However, by partial summation

$$\sum_{n\leqslant\sqrt{x}} d_k(n)\,\mathrm{Li}\left[\frac{x}{n}\right] = \mathrm{Li}(\sqrt{x})\,D_k(\sqrt{x}) + \int_1^{\sqrt{x}} \frac{D_k(t)}{\log(x/t)}\,\frac{Dx}{t^2}\,dt$$

$$= \mathrm{Li}(\sqrt{x})\,D_k(\sqrt{x}) + \frac{x}{\log x}\int_1^{\sqrt{x}} \frac{\Delta_k(t)}{t^2}\left[1 - \frac{\log t}{\log x}\right]^{-1} dt$$

$$+ x\int_1^{\sqrt{x}} \frac{1}{t\log x/t}\sum_{j=0}^{k-1} a_j^{(k)}\,(\log t)^{k-1-j}\,dt.$$

Now

$$\frac{x}{\log x}\int_1^{\sqrt{x}} \frac{\Delta_k(t)}{t^2}\left[1 - \frac{\log t}{\log x}\right]^{-1} dt = \frac{x}{\log x}\int_1^{\sqrt{x}} \frac{\Delta_k(t)}{t^2}\left\{1 + \frac{\log t}{\log x}\right.$$

$$\left. + \dots + \left[\frac{\log t}{\log x}\right]^{N-1} + O\left[\left[\frac{\log t}{\log x}\right]^N\right]\right\} dt$$

$$= \frac{x}{\log x}\sum_{i=0}^{N-1} \frac{1}{(\log x)^i}\int_1^{\infty} \frac{\Delta_k(t)(\log t)^i}{t^2}\,dt + O(x(\log x)^N).$$

Since $\Delta_k(x) = O[x^{1-1/k}]$ Further on writing u for x/t, we get

$$x\int_1^{\sqrt{x}} \frac{1}{t\log x/t}\sum_{j=0}^{k-1} a_j^{(k)}(\log t)^{k-1-j}\,dt = x\sum_{j=0}^{k-1} a_j^{(k)}\int_{\sqrt{x}}^{x} \frac{\{\log(x(u)\}^{k-1-j}}{x/u\log u}$$

$$\times\ \frac{x}{u^2}\,du$$

$$= x\sum_{j=0}^{k-1} a_j^{(k)}\sum_{r=0}^{k-1-j}(-1)^r\left[\begin{matrix}k-1-j\\ l\end{matrix}\right](\log x)^{k-1-j-r}\int_{\sqrt{x}}^{x}\frac{(\log u)^{r-1}}{u}\,du$$

$$= x\sum_{j=0}^{k-1} a_j^{(k)}\left\{(\log x)^{k-1-j}(\log 2) + \sum_{r=1}^{k-1-j}(-1)^r\left[\begin{matrix}k-1-j\\ r\end{matrix}\right]\right.$$

$$\left.\times\ (\log x)^{k-1-r}\,\frac{(1-2^{-r})(\log x)^r}{r}\right\} = x\sum_{j=0}^{k-1} F_j(\log x)^{k-1-j}$$

for certain constants F_j, $0 \leqslant j \leqslant k-1$. Thus with a large C, from (4.2) through (4.8) and the above, we obtain the theorem.

Remark : 4.1: The case $k = 2$ of Theorem 4.1 was earlier attempted by De Koninck and Mercier[6]. See also (De Koninck and Ivić[4], Chapter 9). However, there is a slip, as pointed out by Sitaramachandrarao[16]. Recently, Ivić[14], in addition to correcting this, gave an analytic proof of Theorem 4.1 and several other results. Also, Ivić raised the problem of proving Theorem 4.1 by an elementary method.

We state the following results that can be proved by the methods of this paper. The set of constants A, B, C, D and $E_0, E_1, \ldots, E_{N-1}$ may vary with the asymptotic formula. For each positive integer N, we have

$$\sum_{n\leqslant x} 2^{\omega}(n)\,\omega(n) = Ax(\log x)(\log\log x) + Bx\log x + Cx\log\log x$$

$$+ Dx + x\sum_{i=0}^{N-1}\frac{E_i}{(\log x)^i} + O\left[\frac{x}{(\log x)^N}\right]$$

$$\sum_{n\leqslant x}\mu(n)\,\omega(n) = x\sum_{i=2}^{N}\frac{E_{i-1}}{(\log x)^i} + O\left[\frac{x}{(\log x)^{N+1}}\right]$$

$$\sum_{n\leqslant x}\phi(n)\,\omega(n) = Ax^2\log\log x + Bx^2 + x^2\sum_{i=1}^{N-1}\frac{E_i}{(\log x)^i}$$

$$+ O\left[\frac{x^2}{(\log x)^N}\right]$$

and

$$\sum_{n\leqslant x}\sigma(n)\,\omega(n) = Ax^2\log\log x + Bx^2 + x^2\sum_{i=1}^{N-1}\frac{E_i}{(\log x)^i}$$

$$+ O\left[\frac{x^2}{(\log x)^N}\right].$$

Some of these also appear in Ivić[14]. Finally, we note that for estimating sums of the form $\sum_{n\leqslant x} f(n)\,g(n)$ where $f(n)$ is multiplicative and $g(n)$ is additive, the method of this paper can be used when $g(n)$ is an arbitrary additive function while it appears that the analytic method outlined in Ivić[14] can be used only when $g(n)$ is a prime-independent additive function.

References

1. K. Alladi, *J. Numb. Theory* 9 (1977), 436-51.
2. K. Alladi and P. Erdös. *Pac. J. Math.* 71 (1977), 275-94.
3. N. G. De Bruijn, *Indag. Math.* 13 (1951), 50-60 and 28 (1966), 239-47.

4. J. M. De Koninck and A. Ivić, *Topics in Arithmetical Functions.* Notás de Matematica 72, North Holland, Amsterdam, 1980.
5. J. M. De Koninck and A. Ivić, *Arch. Math.* 43 (1984), 37-43.
6. J. M. De Koninck and A. Mercier, *Can Math Bull.* 20 (1977), 77-80
7. J. M. De Koninck and A. Mercier. Les fonctions arithmétiques et le plus grand facteur premier. *Acta Arithmetica*, in print.
8. J. M. De Koninck and R. Sitaramachandrarao, Sums involving the largest prime divisor of an integer, *Acta Arith.* in print.
9. H. Delange, *Acta Arith.* 19 (1971), 105-46.
10. P. Erdös and A. Ivić, *Studia Scien. Math. Hungarica* 15 (1980), 183-99.
11. P. Erdös, A. Ivic and C. Pomerance, *Glagnik Math.* 2 (1986), 27-44.
12. A. Ivić, *Arch. Math.* 36 (1981), 57-61.
13. A. Ivić, *The Riemann Zeta function*, John Wiley and Sons, New York, 1985, Chap. 14.
14. A. Ivić, Sums of products of certain arithemetical functions, Submitted.
15. A. Ivić and C. Pomerance, Estimates of certain sums involving the largest prime factor of an integer, *Coll. Math. Soc. Janos Bolgai* 34, *Topics in Classical Number Theory*, North Holland, Amsterdum.
16. R. Sitaramachandrarao, Arithmetical functions in short intervals, submitted.
17. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Clarendon Press, Oxford, 1951.