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SOME RANDOMLY SELECTED ARITHMETICAL SUMS

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1. Introduction

Let $p_1(n)<p_2(n)<\ldots<p_\omega(n)$ be the sequence of distinct prime divisors of n; that is,

$$n=\prod_{j=1}^{\omega} p_j^{\alpha_j}(n), \quad n\geqq 2,$$

where $\omega=\omega(n)$ is the number of distinct prime divisors of n.

Recently, several authors investigated the behavior of

$$\sum_{2\leqq n\leqq x}\frac{1}{p_j(n)}=R_j(x)$$

for some specific choices of j. In particular, for $j=1$, or whenever j is preassigned, it is not difficult to show that $R_j(x)\sim c_j x$ with a computable constant c_j. On the other hand, when $j=\omega(n)$, the problem of finding good approximations to $R_j(x)$ becomes very difficult; by refining several earlier results, Ivić and Pomerance [5] found the best known approximation. Quite remarkably, the case of $j=\omega(n)-1$, or $j=\omega(n)-k$ with k fixed, shows no similarity to the case of $j=\omega(n)$, and asymptotic expressions are in fact known (Erdős and Ivić [3]):

$$R_{\omega-k}(x)\sim c_k\frac{x(\log\log x)^{k-1}}{\log x}, \quad k\geqq 1,$$

where c_k is a constant.

In order to obtain an "average type of information" on the magnitude of $p_j(n)$, when j does not belong to the mentioned cases (i.e. either j is fixed or $j=\omega-k$ with k fixed), we set up the following probabilistic approach. For every integer $n\leqq x$, pick one $p(n)$ of its prime divisors $p_j(n)$ with equal probabilities (hence $p(n)=p_j(n)$ with probability $1/\omega(n)$), and consider the sums

$$R(x)=\sum_{2\leqq n\leqq x}\frac{1}{p(n)}. \tag{1}$$

Here and in what follows we assume that x is an integer. Evidently, there are $\omega(2)\omega(3)\ldots\omega(x)$ sums of the type in (1), and

$$\sum_{2\leqq n\leqq x}\frac{1}{p_\omega(n)}\leqq R(x)\leqq\sum_{2\leqq n\leqq x}\frac{1}{p_1(n)}.$$

It turns out that "almost all" sums in (1) are asymptotically equal to the same expression, $cx/\log\log x$, indicating that $R_j(x)$ does not vary much with j when j is not an extreme (constant or $\omega-k$ with k constant).

This probabilistic approach to $R_j(x)$ led us to investigate several other arithmetical sums

$$Q(x)=\sum_{n\leqq x} r(n), \tag{2}$$

where $r(n)$ is one randomly selected member of a set A_n associated with n. We establish that $Q(x)$ is asymptotically the same value for "almost all" selections of $r(n)$. For example, if A_n is the set of the reciprocals of the divisors of n, then it turns out that

$$Q(x)\sim\sum_{n\leqq x}\frac{\sigma(n)}{n\tau(n)}, \tag{3}$$

where $\sigma(n)$ is the sum of the divisors of n, and $\tau(n)$ is the number of divisors of n. On the other hand, if A_n is the set of all divisors of n, then (again for almost all selections of $r(n)$ in (2)),

$$Q(x)\sim\sum_{n\leqq x}\frac{\sigma(n)}{\tau(n)}. \tag{4}$$

These results, therefore, give a probabilistic meaning to the arithmetical sums on the right hand sides of (3) and (4), involving well known arithmetical functions.

2. The sum of reciprocals of random prime divisors of n

As in the introduction, $p_1(n)<p_2(n)<\ldots<p_\omega(n)$ denote the distinct prime divisors of n, and we select one $p(n)$ of these prime divisors at random (with equal probabilities). Set

$$R(x)=\sum_{2\leqq n\leqq x}\frac{1}{p(n)}. \tag{5}$$

Note again the total number of sums of the form of (5) is $\omega(2)\omega(3)\ldots\omega(x)$. We shall say that a property holds for almost all sums in (5) if the number $N(x)$ of the sums with the property in question satisfies

$$N(x)/\omega(2)\omega(3)\ldots\omega(x)\to 1$$

as $x\to+\infty$.

THEOREM 1. *For almost all sums in* (5),

$$R(x)=\frac{c_1x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right)$$

where $c_1=\sum 1/p^2$, *the summation being over all primes* p.

The proof is based on the Chebyshev inequality stated below.

LEMMA. *Let* $A_n=\{a_1, a_2, \ldots, a_{f(n)}\}$, $n\geqq 1$, *be a sequence of finite sets. For every* $n\geqq 1$, *pick one member* $r(n)$ *of* A_n *at random with equal probabilities (i.e.* $r(n)=a_j$ *with probability* $1/f(n)$*), and set*

$$Q(x)=\sum_{n\leqq x} r(n). \tag{6}$$

Then the number $N_Q(x)$ *of sums in* (6) *for which*

$$|Q(x)-E|\geqq V^{5/8}$$

where

$$E=\sum_{n\leqq x}\frac{1}{f(n)}\sum_{j=1}^{f(n)} a_j$$

and

$$V=\sum_{n\leqq x}\frac{1}{f(n)}\sum_{j=1}^{f(n)} a_j^2-\sum_{n\leqq x}\left(\frac{1}{f(n)}\sum_{j=1}^{f(n)} a_j\right)^2$$

satisfies

$$N_Q(x)\leqq V^{-1/4}f(1)f(2)\ldots f(x).$$

PROOF. See Galambos [4].

PROOF OF THEOREM 1. With the notations of the lemma,

$$E=\sum_{2\leqq n\leqq x}\frac{1}{\omega(n)}\sum_{p|n}\frac{1}{p}=\sum_{p\leqq x}\frac{1}{p}\sum_{m\leqq x/p}\frac{1}{\omega(pm)}. \tag{7}$$

Clearly

$$\frac{1}{\omega(m)+1}\leqq\frac{1}{\omega(pm)}\leqq\frac{1}{\omega(m)}.$$

Hence

$$\frac{1}{\omega(pm)}=\frac{1}{\omega(m)}+O\left(\frac{1}{\omega(m)^2}\right),$$

where the $O(\ldots)$ is uniform in $m\geqq 2$. Therefore

$$\sum_{p\leqq x}\frac{1}{p}\sum_{m\leqq x/p}\frac{1}{\omega(pm)}=\sum_{p\leqq x}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}+O\left(\sum_{p\leqq x}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)^2}\right). \tag{8}$$

Now

$$\sum_{\sqrt{x}<p\leqq x}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}\leqq\frac{1}{\sqrt{x}}\sum_{p\leqq x}\sum_{m\leqq x/p}1\leqq\frac{1}{\sqrt{x}}\sum_{p\leqq x}\frac{x}{p}= \tag{9}$$

$$=O(\sqrt{x}\log\log x)=O(x/(\log\log x)^2).$$

We recall the estimates

$$\sum_{2\leqq m\leqq x}\frac{1}{\omega(m)}=\frac{x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right) \tag{10}$$

and

$$\sum_{2\leqq m\leqq x}\frac{1}{\omega(m)^2}=O\left(\frac{x}{(\log\log x)^2}\right) \tag{11}$$

proved in De Koninck [1]. Using (10), we have

$$\sum_{p\leqq\sqrt{x}}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}=\sum_{p\leqq\sqrt{x}}\frac{1}{p}\frac{x/p}{\log\log(x/p)}+O\left(\frac{x/p}{(\log\log(x/p))^2}\right). \tag{12}$$

But since, for $p\leqq\sqrt{x}$,

$$\log\left(1-\frac{\log p}{\log x}\right)=O(1),$$

then

$$\frac{1}{\log\log(x/p)}=\frac{1}{\log\log x+\log\left(1-\frac{\log p}{\log x}\right)}=\frac{1}{\log\log x}\frac{1}{1+\frac{\log\left(1-\frac{\log p}{\log x}\right)}{\log\log x}}= \tag{13}$$

$$=\frac{1}{\log\log x}\left(1+O\left(\frac{\log\left(1-\frac{\log p}{\log x}\right)}{\log\log x}\right)\right)=\frac{1}{\log\log x}+O\left(\frac{1}{(\log\log x)^2}\right).$$

Using this in (12) yields

$$\sum_{p\leqq\sqrt{x}}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}=\frac{x}{\log\log x}\sum_{p\leqq\sqrt{x}}\frac{1}{p^2}+O\left(\frac{x}{(\log\log x)^2}\right).$$

Since

$$\sum_{p\leqq\sqrt{x}}\frac{1}{p^2}=\sum_{p}\frac{1}{p^2}+O\left(\frac{1}{\sqrt{x}}\right)=c_1+O\left(\frac{1}{\sqrt{x}}\right),\quad\text{say,}$$

we finally obtain

$$\sum_{p\leqq\sqrt{x}}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}=c_1\frac{x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right). \tag{14}$$

Also, using (11), we have

$$\sum_{p\leqq\sqrt{x}}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)^2}=O\left(\sum_{p\leqq\sqrt{x}}\frac{1}{p}\frac{x/p}{(\log\log(x/p))^2}\right)=O\left(\frac{x}{(\log\log x)^2}\right), \tag{15}$$

because of (13).

Now combining (14) and (15), and using once more (9), (8) becomes

$$\sum_{p\le x}\frac{1}{p}\sum_{m\le x/p}\frac{1}{\omega(pm)} = c_1\frac{x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right). \tag{16}$$

Hence

$$E = c_1\frac{x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right).$$

By similar calculations, we get

$$V = \sum_{2\le n\le x}\frac{1}{\omega(n)}\sum_{p|n}\frac{1}{p^2}-\sum_{2\le n\le x}\left(\frac{1}{\omega(n)}\sum_{p|n}\frac{1}{p}\right)^2 \le \frac{c_2x}{\log\log x},$$

and thus the lemma implies Theorem 1.

3. Random sums related to the divisors of n

Let now $d_1<d_2<\ldots<d_\tau$ be the divisors of n, where $d_j=d_j(n)$ and $\tau=\tau(n)$ is the number of divisors of n. In addition, let $\sigma(n)$ be the sum of the divisors of n. Again, for each $n\ge2$, we pick one $r(n)$ of the divisors d_j, and, with a given function $h(\cdot)$, we define

$$Q(x) = \sum_{2\le n\le x} h(r(n)). \tag{17}$$

Because the number of sums in (17) is $\tau(2)\tau(3)\ldots\tau(x)$, we now say that a property holds for almost all sums in (17) if it holds for N^* sums such that $N^*/\tau(2)\tau(3)\ldots\tau(x)\to1$ as $x\to+\infty$.

Although the basic idea of the computations is the same for a large variety of choices for $h(u)$, we carry out the computations when $h(u)$ is either $1/u$ or u. Their significance is that the major terms in the asymptotic expressions below are familiar arithmetical sums.

THEOREM 2. *For almost all sums in* (17),

(i) $$\sum_{2\le n\le x}\frac{1}{r(n)} = \sum_{2\le n\le x}\frac{\sigma(n)}{n\tau(n)}+O(x^{5/8}) = \frac{c_3x}{\sqrt{\log x}}+O\left(\frac{x}{(\log x)^{3/2}}\right);$$

and

(ii) $$\sum_{2\le n\le x} r(n) = \sum_{2\le n\le x}\frac{\sigma(n)}{\tau(n)}+O(x^{15/8}) = \frac{c_4x^2}{\sqrt{\log x}}+O\left(\frac{x^2}{(\log x)^{3/2}}\right).$$

PROOF. We again use the Chebyshev inequality stated as Lemma in the previous section.

(i) *The case of* $1/r(n)$:

$$E = \sum_{2\leqq n\leqq x}\frac{1}{\tau(n)}\sum_{d|n}\frac{1}{d} = \sum_{2\leqq n\leqq x}\frac{1}{n\tau(n)}\sum_{d|n}\frac{n}{d} = \sum_{2\leqq n\leqq x}\frac{1}{n\tau(n)}\sum_{d|n}d = \sum_{2\leqq n\leqq x}\frac{\sigma(n)}{n\tau(n)}.$$

An asymptotic expression for the last sum above can easily be obtained by using Dirichlet generating functions. As a matter of fact, since

$$\sum_{n=1}^{+\infty}\frac{\sigma(n)/n\tau(n)}{n^s} = \prod_p\left(1+\frac{\frac{1}{2}(1+1/p)}{p^s}+\frac{\frac{1}{3}(1+1/p+1/p^2)}{p^{2s}}+\dots\right) = (\zeta(s))^{1/2}R(s),$$

where $R(s)=O(1)$ for $\operatorname{Re}(s)>\frac{1}{2}$, we have that

$$\sum_{n\leqq x}\frac{\sigma(n)}{n\tau(n)} = \frac{R(1)x}{\sqrt{\log x}}+O\left(\frac{x}{(\log x)^{3/2}}\right). \tag{18}$$

Now,

$$V = \sum_{2\leqq n\leqq x}\frac{1}{\tau(n)}\sum_{d|n}\frac{1}{d^2}-\sum_{2\leqq n\leqq x}\frac{1}{\tau(n)}\left(\sum_{d|n}\frac{1}{d}\right)^2,$$

which can easily be seen to satisfy $V=O(x)$ (more accurate computation is also possible, but this rough estimate suffices). Hence, Lemma concludes the proof of the statement in part (i).

Turning to (ii), E takes the form

$$E = \sum_{2\leqq n\leqq x}\frac{1}{\tau(n)}\sum_{d|n}d = \sum_{2\leqq n\leqq x}\frac{\sigma(n)}{\tau(n)},$$

which, by partial summation in (18), yields the asymptotic formula of (ii) upon observing that the estimate $V=O(x^3)$ is immediate.

Let us conclude by mentioning that several other familiar arithmetical sums do have probabilistic meaning similar to the ones appearing in Theorem 2. For example, if we pick an exponent $\alpha(n)$ at random in the prime factorization $n=\prod p^\alpha$, then, for almost all choices of $\alpha(n)$,

$$\sum_{n\leqq x}\alpha(n) \sim \sum_{2\leqq n\leqq x}\frac{\Omega(n)}{\omega(n)},$$

where $\Omega(n)$ is the total number of prime divisors of n. This latter sum has been investigated in much detail in De Koninck and Ivić [2], yielding

$$\sum_{n\leqq x}\alpha(n) \sim x,$$

for almost all sums on the left hand side.

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