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ON THE DISTANCE BETWEEN CONSECUTIVE DIVISORS OF AN INTEGER

BY

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ABSTRACT. Let $\omega(n)$ denote the number of distinct prime divisors of a positive integer n. Then we define $h:\mathbb{N} \to \mathbb{R}$ by $h(n) = 0$ if $\omega(n) \leq 1$ and $h(n) = \Sigma_{i=2}^{r} 1/(q_i - q_{i-1})$ if $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r}$, where $q_1 < q_2 < \ldots < q_r$ are primes and $r \geq 2$. Similarly denote by $\tau(n)$ the number of divisors of n and let $H:\mathbb{N} \to \mathbb{R}$ be defined by $H(n) = \Sigma_{i=2}^{\tau(n)} 1/(d_i - d_{i-1})$, where $1 = d_1 < d_2 < \ldots d_{\tau(n)} = n$ are the divisors of n. We prove that there exists constants A and B such that $\Sigma_{n \leq x} h(n) = Ax + O(x(\log \log x)(\log x)^{-1})$ and $\Sigma_{n \leq x} H(n) = Bx + O(x(\log x)^{-1/3})$.

§1. **Introduction**. A natural way to estimate the average distance between the prime divisors of an integer $n = q_1^{\alpha_1} \ldots q_r^{\alpha_r}$ $(r \geq 2)$ is to study the arithmetical function

$$f(n) = \frac{1}{r-1} \sum_{i=2}^{r} (q_i - q_{i-1}) = \frac{P(n) - p(n)}{\omega(n) - 1}. \tag{1.1}$$

Here $q_1 < \ldots < q_r$ are the prime divisors of n, and $P(n)$, $p(n)$, $\omega(n)$ denote respectively the largest prime divisor of n, the smallest prime divisor of n and the number of distinct prime divisors of n. Using a result of J.-M. De Koninck and A. Ivić [1], it follows easily that the average order of $f(n)$ is $cn/\log n$, where $c > 0$ is an absolute constant.

We now introduce another arithmetical function, which also provides information about the distance between the distinct prime divisors of n. We shall denote this function by $h(n)$ and define it as

$$h(n) = \begin{cases} 0 & \text{if } \omega(n) \leq 1, \\ \displaystyle\sum_{i=2}^{\omega(n)} \frac{1}{q_i - q_{i-1}}, & \text{if } n = q_1^{\alpha_1} \ldots q_r^{\alpha_r}, r \geq 2, \\ & \quad q_1 < \ldots < q_r \text{ primes.} \end{cases} \tag{1.2}$$

In many ways this function is more complicated to estimate than $f(n)$, and in Theorem 1 below we shall show that there exists an absolute constant $A > 0$ such that

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$$\lim_{x\to\infty} x^{-1} \sum_{n\le x} h(n) = A.$$

An interesting, yet difficult problem is to determine the maximal order of $h(n)$. One has trivially

$$h(n) \le \omega(n) \ll \log n/\log\log n.$$

Nevertheless $h(n)$ can be fairly large for some n, e.g.

$$\frac{\log N_k}{(\log\log N_k)^2} \ll h(N_k) \ll \frac{\log N_k \log\log\log N_k}{(\log\log N_k)^2}, \tag{1.3}$$

where N_k is the product of the first k primes. This follows from the work of P. Erdös and A. Rényi [3], and the lower bound in (1.3) can be obtained simply as follows. If p_n denotes the n-th prime, then by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} k - 1 &= \sum_{2\le n\le k} \left(\frac{p_n - p_{n-1}}{p_n - p_{n-1}}\right)^{1/2} \le \Big(\sum_{2\le n\le k} (p_n - p_{n-1})\Big)^{1/2} \{h(N_k)\}^{1/2} \\ &= (p_k - 2)^{1/2}\{h(N_k)\}^{1/2}. \end{aligned} \tag{1.4}$$

But by the prime number theorem $p_k \sim k \log k$, $k \sim \log N_k/\log\log N_k$ as $k \to \infty$, hence (1.4) gives the lower bound in (1.3).

It seems equally interesting to study the analogues of (1.1) and (1.2) when one considers not only prime divisors of n, but all possible divisors of n. Thus if $1 = d_1 < d_2 < \ldots < d_{\tau(n)}$ denote the consecutive divisors of n, where $\tau(n)$ is the number of divisors of n, then for $n \ge 2$ one may define

$$F(n) = \frac{1}{\tau(n) - 1} \sum_{i=2}^{\tau(n)} (d_i - d_{i-1}) = \frac{n-1}{\tau(n) - 1} \tag{1.5}$$

as the average distance between the divisors of n. Using Theorem 1.2 of J.-M. De Koninck and A. Ivić [2], it follows readily by partial summation that the average order of $F(n)$ is $dn(\log n)^{-1/2}$ for some absolute $d > 0$.

Therefore it is perhaps more interesting to define the arithmetical function $H(n)$, the analogue of $h(n)$, as

$$H(n) = \begin{cases} 0 & \text{if } n = 1, \\ \displaystyle\sum_{i=2}^{\tau(n)} \frac{1}{d_i - d_{i-1}} & \text{if } n \ge 2, \end{cases} \tag{1.6}$$

where as before $1 = d_1 < \ldots < d_{\tau(n)} = n$ are the consecutive divisors of n. Determining the maximal order of magnitude of $H(n)$ seems to be even more difficult than the corresponding problem for $h(n)$. Our main objective will be to prove that, similarly as $h(n)$, the function $H(n)$ has a finite mean value. The result is contained in Theorem 2 below, and provides information about the distribution of consecutive divisors of an integer. These types of problems have been investigated by several authors, most notable by P. Erdös. A classical conjecture of his states that for almost all n

$$\min_{1 \le i \le \tau(n)-1} d_{i+1}/d_i \le 2.$$

This conjecture, in an even stronger form, has been recently proved by H. Maier and G. Tenenbaum [4].

§2. **Statement of results**

THEOREM 1. *Let $h(n)$ be the arithmetical function defined by (1.2). Then*

$$\sum_{n \le x} h(n) = Ax + O\left(\frac{x \log\log x}{\log x}\right), \tag{2.1}$$

where

$$A = \sum_{p_i < p_j} \frac{1}{(p_j - p_i)p_i p_j} \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right) = 0.299\ldots. \tag{2.2}$$

Here the sum is taken over all pairs of primes (p_i, p_j) such that $p_i < p_j$, while for a fixed pair (p_i, p_j) the product is over all primes p satisfying $p_i < p < p_j$.

For our second result we define (a, b) $(1 \le a < b$ integers) to be a suitable pair if there exists a positive integer n such that a and b are two consecutive divisors of n. Clearly (a, b) is a suitable pair if and only if $a < d < b$ implies $d \nmid [a, b]$, where $[a, b]$ is the lowest common multiple of a and b. Further let $D_{a,b}$ consist of all integers of the form $(d/(d, [a, b]))$ $(a < d < b)$, where $(d, [a, b])$ denotes the greatest common divisor of d and $[a, b]$ and where no element of $D_{a,b}$ is a multiple of another element of $D_{a,b}$. Finally, for a given suitable pair (a, b), write $D_{a,b} = \{d_1, d_2, \ldots, d_r\}$, then we shall denote by $R(a, b)$ the following expression

$$1 - \sum_{1 \le i \le r} \frac{1}{d_i} + \sum_{1 \le i < j \le r} \frac{1}{[d_i, d_j]} - \sum_{1 \le i < j < k \le r} \frac{1}{[d_i, d_j, d_k]} + \ldots + (-1)^r \frac{1}{[d_1, d_2, \ldots, d_r]}$$

where $[c_1, c_2, \ldots, c_s]$ stands for the lowest common multiple of the integers $c_1, c_2, \ldots, c_s$.

THEOREM 2. *Let $H(n)$ be the arithmetical function defined by (1.6). Then*

$$\sum_{n \le x} H(n) = Bx + O(x(\log x)^{-1/3}), \tag{2.3}$$

where

$$B = \sum_{b=2}^{\infty} \sum_{a<b}^{*} \frac{1}{[a, b](b - a)} R(a, b)$$

and the star on the inner sum indicates that the summation runs through suitable pairs (a, b).

§3. **The necessary lemmas**. In this section we shall formulate and prove two technical lemmas, which are necessary for the proof of Theorem 1 and Theorem 2.

LEMMA 1. *Let $p_i > p_j$ be any two fixed primes and let $p_{ij} = \Pi_{p_i<p<p_j} p$, where p denotes primes. Then*

$$\sum_{n \le y, (n, P_{ij}) = 1} 1 = y \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right) + \theta 2^{j-i-1}, \tag{3.1}$$

where $|\theta| \le 1$.

PROOF OF LEMMA 1. The left-hand side of (3.1) is equal to

$$\sum_{n \le y} \sum_{d|(n, P_{ij})} \mu(d) = \sum_{d|P_{ij}} \mu(d)[y/d] = y \sum_{d|P_{ij}} \mu(d)/d$$
$$+ \theta_1 \sum_{d|P_{ij}} 1 = y \prod_{p_i<p<p_j} \left(1 - \frac{1}{p}\right) + \theta 2^{j-i-1},$$

where $|\theta_1| \le 1$, $|\theta| \le 1$.

LEMMA 2. *Let (a, b) be a suitable pair, $g(d) = d/(d, [a, b])$ and $R(a, b)$ be as in §2. Then*

$$\sum_{m \le y, a<d<b \Rightarrow g(d) \nmid m} 1 = yR(a, b) + O(2^{b-a}). \tag{3.2}$$

PROOF OF LEMMA 2. Set $D_{a,b} = \{d_1, \ldots, d_r\}$. Then by the inclusion–exclusion principle, we have

$$\sum_{m \le y, a<d<b \Rightarrow g(d) \nmid m} 1 = [y] - \sum_{1 \le i \le r} \left[\frac{y}{d_i}\right] + \sum_{1 \le i < j \le r} \left[\frac{y}{[d_i, d_j]}\right] - \sum_{1 \le i<j<k \le r}$$
$$\times \frac{y}{[d_i, d_j, d_k]} + \ldots + (-1)^r \frac{y}{[d_1, d_2, \ldots, d_r]} = y\, R(a, b) + 0(2^r),$$

and the result follows since $r \le b - a - 1$.

§4. **Proof of Theorem 1**. Clearly we have

$$\sum_{n \le x} h(n) = \sum_{p_i<p_j \le x} \frac{1}{p_j - p_i} \sum_{n \le x, p_i p_j | n, (n, P_{ij}) = 1} 1, \tag{4.1}$$
$$= \sum_{p_i<p_j \le x, p_i p_j \le x} \frac{1}{p_j - p_i} \sum_{m \le x/p_i p_j, (m, P_{ij}) = 1} 1,$$

where in the last sum m denotes a positive integer. Let $w(x)$ denote a function satisfying

$$\log \log x \ll w(x) \ll x^{1/3} \tag{4.2}$$

and which will be suitably chosen a little later. Since we have

$$\sum_{\substack{p_i<p_j\le x\\ p_ip_j\le x,\, p_j-p_i>w(x)}} \frac{1}{p_j-p_i} \sum_{\substack{m\le x/p_ip_j\\ (m,P_{ij})=1}} 1 \ll \frac{1}{w(x)} \sum_{p_ip_j\le x} \left[\frac{x}{p_ip_j}\right]$$
$$\le \frac{1}{w(x)} \sum_{p_ip_j\le x} \left(\frac{x}{p_ip_j}+1\right) \ll \frac{x}{w(x)} (\log\log x)^2,$$

this means that (4.1) gives

$$\sum_{n\le x} h(n) = \sum_{\substack{p_i<p_j\le x\\ p_ip_j\le x,\, p_j-p_i\le w(x)}} \frac{1}{p_j-p_i} \sum_{\substack{m\le x/p_ip_j\\ (m,P_{ij})=1}} 1 + O\left(\frac{x}{w(x)}(\log\log x)^2\right). \tag{4.3}$$

Next if $p_j > x^{1/2}$ in (4.3), then in view of (4.2) we have

$$p_i > p_j - w(x) > \frac{1}{2}x^{1/2}, \qquad p_ip_j > \frac{1}{2}x,$$

hence

$$\sum_{\substack{p_i<p_j\le x,\, p_ip_j\le x\\ p_j-p_i\le w(x),\, p_j>\sqrt{x}}} \frac{1}{p_j-p_i} \sum_{\substack{m\le x/p_ip_j\\ (m,P_{ij})=1}} 1 \ll \sum_{\substack{p_ip_j\le x,\, p_j>\sqrt{x}\\ 0<p_j-p_i\le w(x)}} \left(\frac{x}{p_ip_j}+1\right)$$
$$\ll \sum_{p_ip_j\le x} 1 \ll x\frac{\log\log x}{\log x}.$$

Therefore the contribution of p_j in (4.3) which satisfy $p_j > x^{1/2}$ is negligible. Hence using (4.3) and Lemma 1 we obtain

$$\sum_{n\le x} h(n) = x \sum_{\substack{p_i<p_j\le x,\, p_ip_j\le x\\ p_j-p_i\le w(x),\, p_j\le\sqrt{x}}} \frac{1}{(p_j-p_i)p_ip_j} \prod_{p_i<p<p_j}\left(1-\frac{1}{p}\right) \tag{4.4}$$
$$+ O\left(\sum_{\substack{p_i<p_j\le x,\, p_ip_j\le x\\ p_j-p_i\le w(x),\, p_j\le\sqrt{x}}} \frac{2^{j-i}}{p_j-p_i}\right) + O\left(x\frac{\log\log x}{\log x}\right),$$

if we choose

$$w(x) = \frac{1}{2}\log x\cdot\log\log x. \tag{4.5}$$

To estimate the first O-term in (4.4) we use the Brun–Titchmarsh inequality (see e.g. H. L. Montgomery [5], Theorem 4.4)

$$\pi(M+N) - \pi(M) \le \frac{2N}{\log N}, \qquad (M, N \ge 2)$$

This allows us to write

$$j - i = \pi(p_j) - \pi(p_i) \leq \frac{2(p_j - p_i)}{\log (p_j - p_i)} \leq \frac{1}{2}\frac{w(x)}{\log w(x)} \leq \frac{1}{2}\log x,$$

provided that $j - i \geq 2$. Estimating trivially the terms with $j = i + 1$ we obtain that the first error term in (4.4) is

$$\ll 2^{(1/2)\log x} \sum_{p_j \leq x^{1/2}} \sum_{n \leq p_j - 2} \frac{1}{p_j - n} \ll 2^{(1/2)\log x}\pi(x^{1/2}) \log x \ll x^{9/10}.$$

Thus to finish the proof of Theorem 1 it remains to show that the error made in replacing the first expression on the right-hand side of (4.4) by Ax is $\ll x(\log\log x)/(\log x)$. In view of the arguments which lead to (4.3) and (4.4) it will be sufficient to estimate the expression

$$R = \sum_{p_i < p_j, p_j > x} \frac{1}{(p_j - p_i)p_i p_j} \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right).$$

Using partial summation and the prime number theorem we obtain

$$(4.6) \qquad R \leq \sum_{p_i < p_j, p_j > x} \frac{1}{(p_j - p_i)p_i p_j} = \sum_{p_j > x} \frac{1}{p_j^2} \sum_{p_i < p_j} \left(\frac{1}{p_j - p_i} + \frac{1}{p_i}\right)$$
$$\ll \sum_{p_j > x} \frac{\log p_j}{p_j^2} \ll \frac{1}{x}.$$

This completes the proof of Theorem 1, and (4.6) shows that the value of the constant A in (2.1) is finite. A numerical calculation shows that $A = 0.299\ldots$.

§5. **Proof of Theorem 2.** Let Σ^* denote summation over suitable pairs (a, b), which were defined in §2. We have

$$(5.1) \qquad \sum_{n \leq x} H(n) = \sum_{2 \leq n \leq x} \sum_{i=2}^{\tau(n)} \frac{1}{d_i - d_{i-1}} = \sum_{a<b, b \leq x-1}^{*} \frac{1}{b-a} \sum_{\substack{n \leq x, a|n, b|n \\ a<d<b => d \nmid n}} 1$$
$$= \sum_{a<b, b \leq x-1}^{*} \frac{1}{b-a} \sum_{\substack{[a,b]m \leq x \\ a<d<b => d \nmid [a,b]m}} 1 = \sum_{a<b, b \leq x-1}^{*} \frac{1}{b-a} \sum_{m \leq x/[a,b]}' 1,$$

where the dash in the last sum indicates that we restrict the sum to positive integers m for which the condition

$$a < d < b => \frac{d}{(d, [a, b])} \nmid m$$

is satisfied. Let now $u(x)$, $v(x)$ be two functions which satisfy

$$(5.2) \qquad \log^{\epsilon} x \ll u(x) \leq \tfrac{1}{2}v(x) \ll \log^{1-\epsilon} x,$$

and which will be suitably determined a little later. We shall show first that the terms in (5.1) for which $b - a > u(x)\log x$ are small. This follows from

$$(5.3)\qquad \sum_{2\le n\le x}\ \sum_{i=2,\,d_i-d_{i-1}>u(x)\log x}^{\tau(n)} \frac{1}{d_i - d_{i-1}} \ll \frac{1}{u(x)\log x}\sum_{n\le x}\tau(n) \ll \frac{x}{u(x)}.$$

The second step in the proof is to show that we can also neglect those terms in (5.1) for which $b > v(x)\log x$. Indeed in view of (5.2) and $ab = [a, b](a, b)$ we have

$$(5.4)\qquad \sum^{*}_{\substack{v(x)\log x<b\le x-1\\ a<b,\,b-a\le u(x)\log x}} \frac{1}{b-a}\sum'_{m\le x/[a,b]} 1 \le x \sum_{v(x)\log x<b\le x}\ \sum_{b-u(x)\log x<a<b}$$
$$\times \frac{(a, b)}{(b-a)ab} \le 2x \sum_{v(x)\log x<b\le x}\ \sum_{b-u(x)\log x<b} \frac{(a, b)}{(b-a)b^2}$$
$$\ll xu(x)\log x \sum_{b>v(x)\log x}\frac{1}{b^2} \ll x\frac{u(x)}{v(x)},$$

since $(a, b) \le b - a$ if $1 \le a < b$. Thus from (5.1), (5.3) and (5.4) it follows that

$$(5.5)\qquad \sum_{n\le x} H(n) = \sum_{b\le v(x)\log x}\ \sum^{*}_{a<b,\,b-a\le u(x)\log x}\frac{1}{b-a}\sum'_{m\le x/[a,b]} 1$$
$$+ O\left(\frac{x}{u(x)}\right) + O\left(x\frac{u(x)}{v(x)}\right).$$

Further we shall show that in the sum over a we may suppose that $b - a \le \log x$, and not only $b - a \le u(x)\log x$. Namely we have

$$(5.6)\qquad \sum_{2\le n\le x}\ \sum_{\substack{i=2,\,d_i-d_{i-1}>\log x\\ d_i\le v(x)\log x}}^{\tau(n)} \frac{1}{d_i-d_{i-1}} \ll x\frac{v(x)}{\log x}$$

To obtain this estimate let $\alpha(n)$ for each n counted in (5.6) denote the number of i's such that both $d_i - d_{i-1} > \log x$ and $d_i \le v(x)\log x$ hold. Then obviously we must have $\alpha(n) \le v(x)$, and the left-hand side of (5.6) is

$$\le \sum_{2\le n\le x}\frac{1}{\log x}\sum_{\substack{i=2,\,d_i-d_{i-1}>\log x\\ d_i\le v(x)\log x}}^{\tau(n)} 1 \le \frac{1}{\log x}\sum_{2\le n\le x}\alpha(n) \ll x\frac{v(x)}{\log x}.$$

Thus we obtain

$$(5.7)\qquad \sum_{n\le x} H(n) = \sum_{b\le v(x)\log x}\ \sum^{*}_{\substack{a<b\\ b-a\le\log x}}\frac{1}{b-a}\sum'_{m\le x/[a,b]} 1$$
$$+ O\left(\frac{x}{u(x)}\right) + O\left(\frac{xu(x)}{v(x)}\right) + O\left(\frac{xv(x)}{\log x}\right).$$

Now we shall simplify the triple sum in (5.7) by using Lemma 2. We have

$$\sum_{m \le x/[a,b]}' 1 = \frac{x}{[a, b]} R(a, b) + O(2^{b-a}), \tag{5.8}$$

and

$$\sum_{b \le v(x)\log x} \ \sum_{a < b, b-a \le \log x}^{*} \frac{2^{b-a}}{b-a} \le 2^{\log x} v(x) \log^2 x \ll x/u(x). \tag{5.9}$$

Hence we may write

$$\sum_{n \le x} H(n) = x \sum_{b \le v(x)\log x} \ \sum_{a<b, b-a \le \log x}^{*} \frac{1}{(b-a)[a, b]} R(a, b) + O\left(\frac{x}{u(x)}\right) + O\left(x\frac{u(x)}{v(x)}\right) + O\left(x\frac{v(x)}{\log x}\right). \tag{5.10}$$

Noting that $R(a, b)$ denotes the density of integers with the constraints given in Lemma 2, we obtain that $0 \le R(a, b) \le 1$ for all a, b. Thus Theorem 2 will follow from (5.10) with a suitable choice of $u(x)$ and $v(x)$ if we can prove

$$\sum_{b > v(x)\log x} \ \sum_{a<b}^{*} \frac{1}{(b-a)[a, b]} + \sum_{b=2}^{\infty} \ \sum_{b-a>\log x, a<b}^{*} \frac{1}{(b-a)[a, b]} \ll \frac{(\log\log x)^2}{\log x} + \frac{1}{v(x)}. \tag{5.11}$$

Recalling that $[a, b](a, b) = ab$ it is seen that the first double sum in (5.11) does not exceed

$$\sum_{b > v(x)\log x} \frac{1}{b} \sum_{a<b, b-a \le \log x} \frac{1}{a} + \sum_{b > v(x)\log x} \ \sum_{a<b, b-a>\log x} \frac{(a, b)}{(b-a)ab}.$$

Now we have

$$\sum_{b > v(x)\log x} \frac{1}{b} \sum_{a<b, b-a \le \log x} \frac{1}{a} \ll \sum_{b > v(x)\log x} \frac{1}{b} \int_{b-\log x}^{b} \frac{dt}{t}$$
$$\ll \sum_{b > v(x)\log x} \frac{1}{b} \log \frac{1}{1 - \dfrac{\log x}{b}} \ll \log x \sum_{b > v(x)\log x} \frac{1}{b^2} \ll \frac{1}{v(x)},$$

so that (5.11) will follow from

$$\Sigma = \sum_{b=2}^{\infty} \ \sum_{a<b, b-a>\log x} \frac{(a, b)}{(b-a)ab} \ll \frac{(\log\log x)^2}{\log x}. \tag{5.12}$$

If we set $b - a = m$, then

$$\sum \ll \sum_{m > \log x} \frac{1}{m} \sum_{b > m} \frac{(b - m, b)}{(b - m)b} = \sum_{m > \log x} \frac{1}{m} S_m,$$

say. But for each $m \geq 1$

$$S_m = \sum_{d|m} \sum_{b > m, (b,m) = d} \frac{(b - m, b)}{(b - m)b} = \sum_{d|m} d \sum_{b > m, (b,m) = d} \frac{1}{(b - m)b}.$$

Hence setting $b = Bd$, $m = Md$, $(B, M) = 1$, we obtain

$$\begin{aligned} S_m &= \sum_{d|m} d \sum_{B > M, (B,M) = 1} \frac{1}{(Bd - Md)Bd} \\ &= \sum_{d|m} \frac{1}{Md} \sum_{B > M, (B,M) = 1} \left(\frac{1}{B - M} - \frac{1}{B}\right) \\ &= \frac{1}{m} \sum_{d|m} \sum_{B > M, (B,M) = 1} \frac{1}{B} \ll \frac{\tau(m) \log m}{m} \end{aligned}$$

Therefore using the elementary estimate

$$\sum_{n \leq x} \tau(n) = x \log x + 0(x)$$

and partial summation, we obtain

$$\sum \ll \sum_{m > \log x} \frac{\tau(m) \log m}{m^2} \ll \frac{(\log \log x)^2}{\log x},$$

hence (5.11) follows. Collecting together all the estimates we finally obtain

$$\begin{aligned} \sum_{n \leq x} H(n) &= Bx + O\left(\frac{x}{u(x)}\right) + O\left(\frac{xu(x)}{v(x)}\right) + O\left(\frac{xv(x)}{\log x}\right) + O\left(\frac{x(\log \log x)^2}{\log x}\right) \\ &= Bx + O(x(\log x)^{-1/3}), \end{aligned}$$

if we choose

$$u(x) = \log^{1/3} x, \; v(x) = \log^{2/3} x.$$

This choice satisfies (5.2), hence we obtain (2.3) with B given by (2.4). A numerical calculation shows that $B = 1.77\ldots$.

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