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RECIPROCALS OF CERTAIN LARGE ADDITIVE FUNCTIONS

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1. **Introduction and statement of results**

Let $\beta(n)=\sum_{p\mid n} p$ and $B(n)=\sum_{p^\alpha\| n}\alpha p$ denote the sum of distinct prime divisors of n and the sum of all prime divisors of n respectively. Both $\beta(n)$ and $B(n)$ are additive functions which are in a certain sense large (the average order of $B(n)$ is $\pi^2 n/(6\log n)$, [1]). For a fixed integer m the number of solutions of $B(n)=m$, is the number of partitions of m into primes, while the number of solutions of $\beta(n)=m$, $\mu^2(n)=1$ is the number of partitions of m into distinct primes. There is a certain analogy between the relation of $\beta(n)$ to $B(n)$ and the relation of the well-known additive functions $\omega(n)=\sum_{p\mid n}1$ and $\Omega(n)=\sum_{p^\alpha\| n}\alpha$. Asymptotic estimates of $B(n)$ were investigated in [1], revealing the connection between $B(n)$ and large prime factors of n. In this paper we turn our attention to sums involving reciprocals of $\beta(n)$ and $B(n)$. We shall prove the following theorems:

THEOREM 1. *For any $\varepsilon>0$ and $x\ge x_0(\varepsilon)$,*

$$(1)\qquad x\exp(-(2+\varepsilon)(\log x\cdot\log\log x)^{1/2})\le\sum_{2\le n\le x}1/B(n)$$

$$\le\sum_{2\le n\le x}1/\beta(n)\le x\exp(-(\tfrac{1}{2}-\varepsilon)(\log x\cdot\log\log x)^{1/2}).$$

THEOREM 2. *There exist positive constants $C_1, C_2>0$ such that*

$$(2)\qquad \sum_{2\le n\le x}B(n)/\beta(n)=x+O(x\exp(-C_1(\log x\cdot\log\log x)^{1/2})),$$

$$(3)\qquad \sum_{2\le n\le x}\beta(n)/B(n)=x+O(x\exp(-C_2(\log x\cdot\log\log x)^{1/2})).$$

THEOREM 3.

$$(4)\qquad \sum_{n\le x}{}'\,1/(B(n)-\beta(n))=Cx+O(x^{1/2}\log x),$$

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where

$$C=\int_0^1 (F(t)-6\pi^{-2})t^{-1}\,dt, \qquad F(t)=\prod_p \left(1+\sum_{k=2}^{\infty}(t^{p(k-1)}-t^{p(k-2)})p^{-k}\right), \tag{5}$$

and $\sum'$ *denotes summation over* $n\le x$ *such that* $B(n)\ne\beta(n)$.

2. **Proofs**

We first prove the lower bound in (1). Let

$$A_k=\{n \mid (n\le x)\wedge(\mu^2(n)=1)\wedge(p(n)\le x^{1/k})\}.$$

where we shall use $p(n)$ to denote the largest prime factor of n, x will be sufficiently large and $k=(\log x/\log\log x)^{1/2}$. If n is a product of k different primes each not exceeding $x^{1/k}$, then $n\in A_k$. There at least $U=3kx^{1/k}/(4\log x)$ primes not exceeding $x^{1/k}$, which means

$$\sum_{n\in A_k} 1 \ge \binom{U}{k}=\frac{U(U-1)\cdots(U-k+1)}{k!}\ge(\tfrac{2}{3}U)^k/k!, \tag{6}$$

since $U-k+1\ge 2U/3$ for x sufficiently large. From Stirling's formula or by induction it is seen that $(k/2)^k>k!$ for $k\ge 6$, which when combined with (6) gives

$$\sum_{n\in A_k} 1\ge x\log^{-k} x. \tag{7}$$

Now for $n\in A_k$ we have $B(n)=\beta(n)\le p(n)\omega(n)\ll\dfrac{x^{1/k}\log x}{\log\log x}$, hence

$$\begin{aligned}\sum_{2\le n\in A_k} 1/B(n)&=\sum_{2\le n\in A_k} 1/\beta(n)\gg x^{-1/k}\log^{-1} x\sum_{n\in A_k} 1\\ &\ge x^{1-1/k}\log^{-k-1} x = x\exp(-2(\log x\cdot\log\log x)^{1/2})\log^{-1} x,\end{aligned} \tag{8}$$

which proves the lower bound in (1). To prove the upper bound in (1) write

$$\begin{aligned}\sum_{2\le n\le x} 1/\beta(n)&=\sum_{2\le n\le x, p(n)\le y} 1/\beta(n)+\sum_{n\le x, p(n)>y} 1/\beta(n)\\ &\le \sum_{2\le n\le x, p(n)\le y} 1 + y^{-1}\sum_{n\le x, p(n)>y} 1\le \psi(x,y)+xy^{-1}\end{aligned} \tag{9}$$

where $y=y(x)>2$ will be suitably chosen in a moment. For the function

$$\psi(x,y)=\sum_{n\le x, p(n)\le y} 1$$

we use the following estimate of [2]:

$$\psi(x,y)<c_3x\log^2 y\cdot\exp(-\alpha(\log\alpha+\log\log\alpha-c_4)), \tag{10}$$

where c_3 and c_4 are some positive, absolute constants, $\lim_{x\to\infty} y=\infty$ and

$$3<\alpha=\log x/\log y<4y^{1/2}/(\log y). \tag{11}$$

Now we choose

$$y=\exp((\log x\cdot\log\log x)^{1/2}). \tag{12}$$

Then (11) is satisfied for $x\geq x_0$ and

$$\psi(x,y)\ll_\varepsilon x\exp(-(\tfrac{1}{2}-\varepsilon)(\log x\cdot\log\log x)^{1/2}), \tag{13}$$

where $\ll_\varepsilon$ means that the constant implied by the symbol $\ll$ depends on ε only. Substitution in (9) then gives the right-hand side inequality in (1), finishing the proof of Theorem 1.

To prove Theorem 2 it is enough to prove (2), since trivially

$$\sum_{2\leq n\leq x}\beta(n)/B(n)\leq x+O(1), \tag{14}$$

and by the Cauchy–Schwarz inequality we have

$$x^2+O(x)\leq\Big(\sum_{2\leq n\leq x}1\Big)^2\leq\sum_{2\leq n\leq x}B(n)/\beta(n)\sum_{2\leq m\leq x}\beta(m)/B(m), \tag{15}$$

so that (2) then implies (3). Let

$$S=\sum_{2\leq n\leq x}B(n)/\beta(n)=S_1+S_2, \tag{16}$$

where in S_1 summation is over $2\leq n\leq x$ such that $B(n)<k\beta(n)$, and in S_2 over $2\leq n\leq x$ such that $B(n)\geq k\beta(n)$, where $k=k(x)$ is a large number which will be suitably chosen later. Note that if $B(n)\geq r\beta(n)$ for some integer $r\geq 2$, then n must be divisible by p^r for some prime p, so that the number of $n\leq x$ for which p^r divides n for some p is $\ll\sum_p xp^{-r}\ll x2^{-r}$. Then we have

$$S_2=\sum_{r\geq k}\;\sum_{2\leq n\leq x,\, r\leq B(n)/\beta(n)<r+1}B(n)/\beta(n)\ll\sum_{r\geq k}x(r+1)2^{-r}\ll x\exp(-C_3k) \tag{17}$$

for some $C_3>0$. To estimate S_1 write

$$S_1=S_1'+S_1''. \tag{18}$$

In S_1'', summation is over $2\leq n\leq x$ such that $B(n)<k\beta(n)$ and n is divisible by p^2 for some prime $p>L$, where $L=L(x)$ is a large number that will be suitably chosen. Thus we obtain

$$S_1''\ll k\sum_{n^2m\leq x,\, n>L}1\ll k\sum_{n>L}xn^{-2}\ll kx/L. \tag{19}$$

If $n=p_1^{a_1}\cdots p_i^{a_i}$ is counted in S_1' then $a_j=1$ for $p_j>L$ and $j=1,\ldots,i$, which implies

$$\begin{aligned}B(n)&=(a_1-1)p_1+\cdots+(a_i-1)p_i+\beta(n)\leq L(a_1+\cdots+a_i-i)+\beta(n)\\&\leq L(\Omega(n)-\omega(n))+\beta(n)\leq L(\log x/\log 2)+\beta(n).\end{aligned} \tag{20}$$

Therefore we have

$$S_1' \le \sum_{n\le x} 1 + L(\log x/\log 2) \sum_{2\le n\le x} 1/\beta(n) \tag{21}$$
$$\le x + O(xL \log x \cdot \exp(-C_4(\log x \cdot \log\log x)^{1/2})),$$

where we have used (1) to estimate $\sum_{2\le n\le x} 1/\beta(n)$. From (16)–(21) we obtain

$$S \le x + O(kx/L) + O(x\exp(-C_3 k)) \tag{22}$$
$$+ O(xL \log x \cdot \exp(-C_4(\log x \cdot \log\log x)^{1/2})).$$

Noting that trivially $S \ge x + O(1)$ and choosing

$$k = (\log x \cdot \log\log x)^{1/2}, \tag{23}$$

$$L = \exp(C_5(\log x \cdot \log\log x)^{1/2}), \qquad C_5 = C_4/2, \tag{24}$$

we obtain (2) from (22).

To prove Theorem 3 we employ an analytical method. Let $0 \le t \le 1$ and observe that $t^{B(n)-\beta(n)}$ is a multiplicative function of n satisfying $t^{B(p^k)-\beta(p^k)} = t^{p(k-1)}$ for $k = 1, 2, \ldots$ and every prime p. Therefore for $\operatorname{Re} s > 1$

$$\sum_{n=1}^{\infty} t^{B(n)-\beta(n)} n^{-s} = \prod_p (1 + p^{-s} + t^p p^{-2s} + t^{2p} p^{-3s} + \cdots) \tag{25}$$
$$= \zeta(s) \prod_p (1 + (t^p - 1)p^{-2s} + (t^{2p} - t^p)p^{-3s} + \cdots) = \zeta(s) G(s, t),$$

where $\zeta(s)$ is the Riemann zeta-function and for $\operatorname{Re} s > \frac{1}{2}$

$$G(s, t) = \sum_{n=1}^{\infty} g(n, t) n^{-s}, \tag{26}$$

and $g(n, t)$ is a multiplicative function of n for which $g(p, t) = 0$ and $|g(p^k, t)| \le 1$ for $k \ge 2$. Therefore uniformly for $0 \le t \le 1$ we have

$$\sum_{n\le x} |g(n, t)| \ll x^{1/2}, \tag{27}$$

and by partial summation we subsequently obtain

$$\sum_{n\le x} t^{B(n)-\beta(n)} = \sum_{n\le x} g(n, t)[x/n] = x \sum_{n\le x} g(n, t)/n + O\Big(\sum_{n\le x} |g(n, t)|\Big) \tag{28}$$
$$= xG(1, t) + O(x^{1/2}),$$

where

$$G(1, t) = \prod_p \Big(1 + \sum_{k=2}^{\infty} (t^{p(k-1)} - t^{p(k-2)})p^{-k}\Big) = F(t),$$

and therefore

$$F(0) = \prod_p (1 - p^{-2}) = 6/\pi^2.$$

Note that $B(n)=\beta(n)$ if and only if n is squarefree. Therefore we have uniformly in t

$$\sum_{n\le x}{}' t^{B(n)-\beta(n)-1} = \sum_{n\le x, B(n)\ne\beta(n)} t^{B(n)-\beta(n)-1}$$

$$= xt^{-1}F(t)+O(x^{1/2}t^{-1})-\sum_{n\le x}\mu^2(n)t^{-1} \tag{29}$$

$$= x(F(t)-6/\pi^2)t^{-1}+O(x^{1/2}t^{-1}).$$

Since $F(0)=6/\pi^2$ the function $(F(t)-6/\pi^2)t^{-1}$ is continuous for $0\le t\le 1$, and we obtain the conclusion of the theorem integrating (29) over t from $\varepsilon(x)=x^{-2/3}$ to 1, since

$$\int_{\varepsilon(x)}^{1} \sum_{n\le x}{}' t^{B(n)-\beta(n)-1}\,dt = \sum_{n\le x}{}' 1/(B(n)-\beta(n))+O(x^{1/3}), \tag{30}$$

$$x\int_0^{\varepsilon(x)} (F(t)-6/\pi^2)t^{-1}\,dt \ll x\varepsilon(x)=x^{1/3}, \tag{31}$$

$$\int_{\varepsilon(x)}^{1} O(x^{1/2}t^{-1})\,dt \ll x^{1/2}\log 1/\varepsilon(x) \ll x^{1/2}\log x. \tag{32}$$

3. Some remarks

It seems probable that the inequalities (1) may be replaced by asymptotic formulae, viz.

$$\log \sum_{2\le n\le x} 1/B(n) \sim \log x - C(\log x\cdot\log\log x)^{1/2}, \qquad x\to\infty, \quad C>0 \tag{33}$$

(and a similar formula with $\beta(n)$ instead of $B(n)$), but we are unable to prove (33). Our results concerning $B(n)$ and $\beta(n)$ may be compared with corresponding results for "small" additive functions $\Omega(n)$ and $\omega(n)$. Utilizing essentially the method of proof of Theorem 3 it was shown in [3] that

$$\sum_{2\le n\le x} 1/\Omega(n) = x/\log\log x + a_2x/(\log\log x)^2+\cdots+a_{N-1}x/(\log\log x)^{N-1} + O(x/(\log\log x)^N), \tag{34}$$

$$\sum_{2\le n\le x} 1/\omega(n) = x/\log\log x + b_2x/(\log\log x)^2+\cdots+b_{N-1}x/(\log\log x)^{N-1} + O(x/(\log\log x)^N), \tag{35}$$

where the a_i's and b_i's are computable constants and N is arbitrary, but fixed.

Similarly [4] contains a proof that

$$\sum_{2\le n\le x} \Omega(n)/\omega(n) = x + c_1x/\log\log x+\cdots+c_{N-1}x/(\log\log x)^{N-1} + O(x/(\log\log x)^N), \tag{36}$$

and the formulae (34)–(36) are further sharpened in [5].

The degree of sharpness of the above formulae is not attained in our theorems concerning $\beta(n)$ and $B(n)$, which is to be expected since $\beta(n)$ and $B(n)$ are much larger functions than $\omega(n)$ and $\Omega(n)$, possessing notably wider fluctuations in size.

It is clear that the method of proof of Theorem 2 would yield (2) and (3) with $B(n)$ and $\beta(n)$ replaced by $B^m(n)$ and $\beta^m(n)$ respectively, where m is a fixed positive integer. Our methods also work in the general case of other large additive functions defined by

$$f(n)=\sum_{p\mid n} h(p), \qquad F(n)=\sum_{p^\alpha \| n} \alpha h(p),$$

where for some fixed $K, \gamma>0$ and a fixed real δ we have

$$h(x)=\exp(K\log^\gamma x\cdot(\log\log x)^\delta).$$

For other results and problems concerning $B(n)$ and $\beta(n)$ the reader is referred to [1].

Closely related to $B(n)$ and $\beta(n)$ is the function $B_1(n)=\sum_{p^\alpha\|n} p^\alpha$. From $B_1(n)\geq\beta(n)$ and the fact that $B_1(n)=B(n)=\beta(n)$ if $n\in A_k$ (the set defined at the beginning of §2) we conclude that the bounds of Theorem 1 hold also for

$$\sum_{2\leq n\leq x} 1/B_1(n).$$

It seems likely that

$$\sum_{2\leq n\leq x} B_1(n)/\beta(n)=(c_1+o(1))x\log\log x \tag{37}$$

and

$$\sum_{2\leq n\leq x} B_1(n)/B(n)=(C+o(1))x, \quad C>0. \tag{38}$$

We can rigorously prove at present only

$$\sum_{2\leq n\leq x} B_1(n)/\beta(n)\geq\tfrac{1}{2}x\log\log x+o(x\log\log x). \tag{39}$$

To see this let $p_1<\cdots<p_k$ be the primes not exceeding x. Suppose $p_i^{l_i}\leq x<p_i^{l_i+1}$ $(i\leq k)$ and define $t_i\geq 1$ by

$$t_ip_i^{l_i}\leq x<(t_i+1)p_i^{l_i}, \tag{40}$$

so that $t_i<p_i$. Then we have

$$S=\sum_{2\leq n\leq x} B_1(n)/\beta(n)>\sum_{i\leq k}\sum_{s\leq t_i} B_1(sp_i^{l_i})/\beta(sp_i^{l_i}), \tag{41}$$

Now $\beta(sp_i^{l_i}) \leq \beta(s) + \beta(p_i^{l_i}) \leq s + p_i \leq t_i + p_i < 2p_i$ and $B_1(sp_i^{l_i}) \geq p_i^{l_i}$, which gives

$$S > \sum_{i\leq k} \sum_{s \leq t_i} p_i^{l_i}/(2p_i) \geq \sum_{i \leq k} t_i p_i^{l_i}/(2p_i) \geq \frac{1}{2} \sum_{i \leq k} (xp_i^{-l_i} - 1)p_i^{l_i - 1}$$

$$\geq \frac{x}{2} \sum_{i \leq k} 1/p_i + O\left(\sum_{i \leq k} p_i^{l_i - 1}\right) \geq \frac{x}{2} \log\log x + o(x \log\log x),$$

since

$$\sum_{p \leq x} 1/p = \log\log x + O(1) \quad \text{and} \quad \sum_{i \leq k} p_i^{l_i - 1} = o(x \log\log x).$$

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