manuscripta math. 30, 329 - 341 (1980)

manuscripta mathematica

SUMS OF RECIPROCALS OF CERTAIN ADDITIVE FUNCTIONS

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We obtain sharp estimates for sums of reciprocals and sums of quotients of certain non-negative integer-valued additive arithmetical functions.

1. Introduction

In this paper we give sharp estimates for the sums $\sum'_{n\le x} 1/f(n)$ and $\sum'_{n\le x} g(n)/f(n)$, where $f(n)$ and $g(n)$ belong to a certain class of non-negative, integer-valued additive arithmetical functions, (here $\sum'$ denotes summation over those n for which $f(n) \neq 0$). In particular, we improve over the following asymptotic formulas proved in [1] and [2], respectively:

$$(1.1) \qquad \sum_{n\le x}{}' 1/\omega(n) = x\sum_{i=1}^{N} \frac{a_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{N+1}}\right),$$

$$(1.2) \qquad \sum_{n\le x}{}' \Omega(n)/\omega(n) = x + x\sum_{i=1}^{N} \frac{b_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{N+1}}\right),$$

where $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime divisors and the number of all prime divisors of n respectively, all the a_i's and b_i's are computable constants, and $N \ge 1$ is an arbitrary fixed integer.

One will observe that $\sum_{i=1}^{N} \frac{a_i}{(\log\log x)^i} = a_1 L(x)$,

where for $x \ge x_0$ the function $L(x)$ is positive, continuous, and for every $c > 0$ has the property

0025-2611/80/0030/0329/$02.60

$$(1.3) \qquad \lim_{x\to\infty} L(cx)\,/\,L(x) = 1 \;.$$

Such functions $L(x)$ are called slowly oscillating (or slowly varying; see [7] for a comprehensive account), and their canonical representation is

$$(1.4) \qquad L(x) = \rho(x) \exp\left(\int_{x_0}^{x} \delta(t)\, t^{-1}\, dt \right) ,$$

where $\rho(x)$ and $\delta(x)$ are continuous for $x \ge x_0$, $\lim_{x\to\infty} \rho(x) = A > 0$ and $\lim_{x\to\infty} \delta(x) = 0$. All slowly oscillating functions that appear in the sequel admit an asymptotic expansion in terms of negative powers of $\log\log x$, that is to say, for every fixed integer $M \ge 1$ there exist constants $A_1, A_2, \ldots, A_{M-1}$ such that

$$L(x) = \frac{1}{\log\log x} + \frac{A_1}{(\log\log x)^2} + \ldots + \frac{A_{M-1}}{(\log\log x)^M} + O\left(\frac{1}{(\log\log x)^{M+1}}\right).$$

2. Theorems and proofs

Theorem 1. Let $f(n)$ be a non-negative, integer-valued additive arithmetical function such that for every prime p, $f(p) = 1$ and $f(p^k) < Ck$ for every $k \ge 2$ and some fixed $C > 0$. Then for every fixed integer $N \ge 1$ there exist computable constants $c_1, \ldots, c_N$ such that

$$(2.1) \qquad \sum_{n\le x}{}' \, 1/f(n) = c_1\, x\, L_1(x) + \ldots + \frac{c_N\, x\, L_N(x)}{\log^{N-1} x} + O\left(\frac{x}{\log^N x}\right),$$

where every $L_j(x)$ $(j = 1, \ldots, N)$ is a slowly oscillating function asymptotic to $1/\log\log x$.

Theorem 2. Let $f(n)$ be a non-negative, integer-valued additive arithmetical function such that for every prime p, $f(p) = 0$, $f(p^2) = 1$, and $0 < f(p^k) < Ck$ for every $k \ge 3$ and some fixed $C > 0$. Then for every fixed integer $N \geq 1$ there exist computable constants $e_0, e_1, \ldots, e_N$ such that

$$(2.2) \qquad \sum_{n\leq x}{}' 1/f(n) = e_0 x + \frac{e_1 x^{1/2} L_1(x)}{\log x} + \ldots + \frac{e_N x^{1/2} L_N(x)}{\log^N x} + O\left(\frac{x^{1/2}}{\log^{N+1} x}\right),$$

where every $L_j(x)$ $(j=1,\ldots,N)$ is a slowly oscillating function asymptotic to $1/\log\log x$.

Theorem 3. Let f and g be two non-negative, integer-valued additive arithmetical functions such that for all primes p and integers $k \geq 2, f(p) = g(p) = 1$ $f(p^k) < Ck$, $g(p^k) < Ck$, where C is a positive constant. Then for every fixed integer $N \geq 1$ there exist computable constants a_j, b_j $(j = 1,\ldots,N)$ such that

$$(1) \quad \sum_{n\leq x}{}' \frac{g(n)}{f(n)} = x\left(a_1 + b_1 L_1(x) + \frac{a_2+b_2L_2(x)}{\log x} + \ldots + \frac{a_N+b_NL_N(x)}{\log^{N-1} x}\right) + O\left(\frac{x}{\log^N x}\right),$$

where every $L_j(x)$ $(j = 1,\ldots,N)$ is a slowly oscillating function asymptotic to $1/\log\log x$.

Proofs. We shall use two deep results of H. Delange (proved in [3] and [4] respectively), which we state here as

Lemma 1. Let $f(n)$ be a non-negative, integer-valued additive arithmetical function such that $f(p) = 1$ for every prime p . Let $\sigma_0(\rho)$ denote for every $\rho \geq 0$ the infimum of real numbers $\sigma > 1/2$ for which

$$(2.3) \qquad \sum_{p,k\geq 2} \rho^{f(p^k)} p^{-k\sigma} < +\infty$$

if this set is non-empty (and $\sigma_0(\rho) = +\infty$ otherwise), and let E be the set of all $\rho \geq 0$ for which $\sigma_0(\rho) < 1$ and R the supremum of the set E (finite or $+\infty$). Then for every fixed integer $N \geq 0$ there exist functions $A_0, A_1, \ldots, A_N$ analytic for $|z| < R$ and continuous for $|z| \leq R$ such that $A_0(0) = A_1(0) = \ldots = A_N(0) = 0$ and

$$(2.4)\qquad \sum_{n\le x} z^{f(n)} = x(\log x)^{z-1}\left(\sum_{j=0}^{N}\frac{A_j(z)}{(\log x)^j} + O\left(\frac{1}{\log^{N+1} x}\right)\right),$$

where for every $\rho > 0$ from E the O-constant is uniform for $|z| \le \rho$.

Lemma 2. Let $f(n)$ be a non-negative, integer-valued additive arithmetical function such that for every prime p, $f(p) = 0$ and $f(p^2) = 1$. For every $\rho \ge 0$ define a multiplicative function h_ρ by

$$(2.5)\qquad h_\rho(p^k) = \begin{cases} \rho^{f(p^k)} + \rho^{f(p^{k-1})} & f(p^k) \ne f(p^{k-1}) \\ 0 & f(p^k) = f(p^{k-1}) \end{cases}$$

and for every $\rho \ge 0$ let $\sigma_0(\rho)$ be the infimum of real numbers $\sigma > 1/3$ for which

$$(2.6)\qquad \sum_{p,\,k\ge 3} h_\rho(p^k)\, p^{-k\sigma} < +\infty$$

if this set is non-empty (and $\sigma_0(\rho) = +\infty$ otherwise). Let further I be the set of $\rho \ge 0$ for which $\sigma_0(\rho) < 1/2$, and R the supremum of I (finite or $+\infty$). Then for every fixed integer $N \ge 0$ there exist functions $F, A_0, A_1, \ldots, A_N$ analytic for $|z| < R$ and continuous for $|z| \le R$ such that $F(0) = 6/\pi^2, A_0(0) = \ldots = A_N(0) = 0$ and

$$(2.7)\qquad \sum_{n\le x} z^{f(n)} = xF(z) + x^{1/2}(\log x)^{z-2}\left(\sum_{j=0}^{N}\frac{A_j(z)}{(\log x)^j} + O\left(\frac{1}{\log^{N+1} x}\right)\right),$$

where for every $\rho > 0$ from I the O-constant is uniform for $|z| \le \rho$.

We begin now the proof of Th. 1 by observing that if f satisfies the hypothesis of Th. 1, then Lemma 1 may be applied with some $R > 1$, and so (2.1) holds uniformly in z for $|z| \le 1$. To see this note that if $\rho \ge 1$

$$\sum_{p,k\ge 2} \rho^{f(p^k)} p^{-k\sigma} \le \sum_{p,k\ge 2} (\rho^C p^{-\sigma})^k = \sum_p (\rho^C p^{-\sigma})^2 / (1-\rho^C p^{-\sigma}) < +\infty$$

for every $\sigma > 1/2$, provided that $\rho^C p^{-\sigma} < 1 - B$ for some fixed $0 < B < 1$. if $\sigma \ge 2/3$ then

$$\rho^C p^{-\sigma} \le \rho^C 2^{-2/3} < 5/6$$

for $\rho < (\frac{5}{6} 2^{2/3})^{1/C}$. Since C is fixed this last number is greater than unity, and thus Lemma 1 applies with some $R > 1$.

Now $f(n) = 0$ for $n \le x$ if $n = 1$ or possibly if n is one of the $O(x^{1/2})$ "square-full" numbers not exceeding x (numbers of the form $n = p_1^{a_1} \dots p_i^{a_i}$ where $a_1 \ge 2, \dots, a_i \ge 2$). If $f(n) \ne 0$, then $f(n) \ge 1$ since f is integer-valued and non-negative. Dividing (2.4) by z and setting $B_j(z) = A_j(z)/z$ we see that for $|z| \le 1$ we have uniformly

$$(2.8) \qquad \sum_{n\le x}{}' z^{f(n)-1} = x(\log x)^{z-1}\left(\sum_{j=0}^{N} \frac{B_j(z)}{(\log x)^j} + O\left(\frac{1}{z(\log x)^{N+1}}\right)\right) .$$

We take now z real and integrate (2.8) from $\varepsilon(x) = x^{-2/3}$ to 1 over z. The left-hand side of (2.8) becomes after integration

$$(2.9) \qquad \sum_{n\le x}{}' 1/f(n) - \sum_{n\le x}{}' (\varepsilon(x))^{f(n)}/f(n) = \sum_{n\le x}{}' 1/f(n) + O(x^{1/3}) ,$$

since $(\varepsilon(x))^{f(n)} \ll \varepsilon(n)$ if $f(n) \ne 0$.

Integrating the right-hand side of (2.8) it is seen that the integral of the error term is

$$(2.10) \qquad \int_{\varepsilon(x)}^{1} x \log^{z-N-2} x \, . \, \frac{dz}{z} \ll x \log^{-N} x .$$

When integrating the main terms on the right-hand side of (2.8) we encounter integrals of the form

$$(2.11)\qquad x(\log x)^{-1-j}\int_{\varepsilon(x)}^{1} B_j(z)\log^z x\,.\,dz = x(\log x)^{-1-j}\int_0^1 B_j(z)\log^z x\,.\,dz + O(x^{1/3})\ ,$$

since $B_j(z) = A_j(z)/z$ is an analytic function for $|z| \le 1$ (because $R > 1$ and $A_j(0) = 0$) , so that for $x \ge x_0$

$$\int_0^{\varepsilon(x)} B_j(z)\log^z x\,.\,dz \ll (\log x)^{\varepsilon(x)}\int_0^{\varepsilon(x)} |B_j(z)|\,dz$$

$$\ll \exp(x^{-2/3}\log\log x)\,.\,\varepsilon(x)\max_{z\in[0,1]}|B_j(z)| \ll \varepsilon(x) = x^{-2/3}\ .$$

Integration by parts gives for every fixed integer $M \ge 1$

$$H_j(x) = \int_0^1 B_j(z)\log^z x\,.\,dz = \frac{B_j(z)\log^z x}{\log\log x}\Bigg|_0^1 + \ldots + (-1)^{M-1}\frac{B_j^{(M)}(z)\log^z x}{(\log\log x)^{M-1}}\Bigg|_0^1$$

$$(2.12)\; + O\left(\int_0^1 \frac{|B_j^{(M+1)}(z)|\log^z x}{(\log\log x)^{M+1}}\,dz\right) = \frac{B_j(1)\log x}{\log\log x} + \ldots + (-1)^{M-1}\frac{B_j^{(M)}(1)\log x}{(\log\log x)^{M}}$$

$$+ O\left(\frac{\log x}{(\log\log x)^{M+1}}\right)\ ,$$

which means that $L_j(x) = H_j(x)/(B_j(1)\log x)$ is a slowly oscillating function asymptotic to $1/\log\log x$ which admits an expansion in terms of negative powers of $\log\log x$. From (2.9) - (2.12) Th. 1 follows with $c_i = B_{i-1}(1)$. Since the functions $A_j(z)$ may be explicitly written, as was done in [3], this means that all the constants c_i are computable.

To prove Th. 2 we use Lemma 2 and exactly the same method of proof again, noting that similarly as before the hypothesis that $f(p^k) < Ck$ assures that $R > 1$ in Lemma 2, so that (2.7) holds uniformly for $|z| \le 1$ if f satisfies the hypothesis of Th. 2. Observe next that $f(n) = 0$ if and only if n is square-free, so that (see [8] for a proof)

$$(2.13) \qquad \sum_{n\le x, f(n)=0} z^{f(n)} = \sum_{n\le x} \mu^2(n) = \frac{6}{\pi^2}x + O(x^{1/2}\exp(-c\,\delta(x))) \;,$$

where c is a positive constant and $\delta(x) = \log^{3/5} x\,.\,(\log\log x)^{-1/5}$. Therefore dividing (2.7) by z we obtain uniformly for $|z| \le 1$

$$(2.14) \qquad \sum_{n\le x}{}' z^{f(n)-1} = x(F(z) - 6/\pi^2)\,z^{-1} + x^{1/2}\log^{-2} x\,.\sum_{j=0}^{N} B_j(z)\log^{z-j} x +$$

$$O(x^{1/2}|z|^{-1}\log^{\mathrm{Re}\, z-N-3} x) + O(x^{1/2}|z|^{-1}\exp(-c\,\delta(x)))\;,$$

where we have set again $B_j(z) = A_j(z)/z$ (though of course $A_j(z)$ of Lemma 1 may be a different function from $A_j(z)$ of Lemma 2).

Now we integrate (2.14) over z from $\varepsilon(x) = x^{-2/3}$ to 1, exactly as was done in the proof of Th. 1. The left-hand side becomes then again

$$(2.15) \qquad \sum_{n\le x}{}' 1/f(n) + O(x^{1/3}) \;,$$

and likewise

$$(2.16) \qquad \int_{\varepsilon(x)}^{1} B_j(x)\log^z x\,.\,dz = \int_0^1 B_j(z)\log^z x\,.\,dz + O(\varepsilon(x))\;,$$

$$(2.17) \qquad \int_0^1 B_j(z)\log^z x\,.\,dz = B_j(1)\log x\,.\,L_j(x)\;,$$

where $L_j(x)$ is a slowly oscillating function asymptotic to $1/\log\log x$. Integrating the error terms on the right-hand side of (2.14) we obtain

$$(2.18)\qquad O(x^{1/2}\log^{-N-1}x) + O(x^{1/2}\log x\,.\,\exp(-c\,\delta(x))) = O(x^{1/2}\log^{-N-1}x)\ ,$$

and finally

$$(2.19)\quad \int_{\varepsilon(x)}^{1} x(F(z) - 6/\pi^2)\,z^{-1}\,dz = x\int_0^1 (F(z) - 6/\pi^2)\,z^{-1}\,dz - x\int_0^{\varepsilon(x)} (F(z) - 6/\pi^2)\,z^{-1}\,dz$$

$$= x\int_0^1 (F(z) - 6/\pi^2)\,z^{-1}\,dz + O(x\,\varepsilon(x)) = e_0 x + O(x^{1/3})\ ,$$

since $(F(z) - 6/\pi^2)\,z^{-1}$ is continuous in $[0,1]$ because $F(0) = 6/\pi^2$. Combining (2.15) - (2.19) we obtain Th. 2 with

$$e_0 = \int_0^1 (F(z) - 6/\pi^2)\,z^{-1}\,dz\ ,\quad e_j = B_{j-1}(1)$$

for $j \ge 1$.

In order to prove Theorem 3, we proceed as follows. Define for every pair (ρ_1,ρ_2) of non-negative real numbers $\sigma_0(\rho_1,\rho_2)$ as the infimum of real numbers $\sigma > 1/2$ for which

$$(2.20)\qquad S = \sum_{p,k\ge 2} \rho_1^{g(p^k)}\,\rho_2^{f(p^k)}\,p^{-k\sigma} < +\infty$$

if this set is non-empty, and $\sigma_0(\rho_1,\rho_2) = +\infty$ otherwise. Let E be the set of non-negative pairs of real numbers (ρ_1,ρ_2) for which $\sigma_0(\rho_1,\rho_2) < 1$.

Suppose now $\rho_1 \ge 1$ and $\rho_2 \ge 1$. Then the hypothesis of the theorem imply

$$S \le \sum_{p,k\ge 2} \rho_1^{Ck}\,\rho_2^{Ck}\,p^{-k\sigma} = \sum_{p} \rho_1^{2C}\,\rho_2^{2C}\,p^{-2\sigma} / (1 - \rho_1^{C}\rho_2^{C}\,p^{-\sigma}) < +\infty$$

for every $\sigma > 1/2$ if $\rho_1^C\rho_2^C\,p^{-\sigma} \le 1 - B$ for some fixed $0 < B < 1$. If $\sigma \ge 2/3$,

$\rho_1 < (\frac{9}{10} 2^{1/3})^{1/C}$, $\rho_2 < (\frac{9}{10} 2^{1/3})^{1/C}$, then we obtain

$$\rho_1^C \rho_2^C < \frac{81}{100} 2^{2/3} \le \frac{81}{100} p^\sigma$$

Since $\frac{9}{10} 2^{1/3} > 1$, this means that $(\rho,\rho) \in E$ for some fixed $\rho > 1$. By the lemma of Delange [2], p. 108, this implies that

$$G(s,z,u) = \prod_p (1 + \sum_{k=1}^{\infty} z^{g(p^k)} u^{f(p^k)} p^{-ks}) (1 - p^{-s})^{zu}$$

is a well-defined function for $|z| \le \rho$, $|u| \le \rho$, $\sigma = \operatorname{Re} s > \sigma_0(\rho,\rho)$, analytic in s . Proceeding further similarly as was done by Delange in [3], we obtain that for every fixed integer $N \ge 0$

$$\sum_{n\le x} z^{g(n)} u^{f(n)} = x(\log x)^{zu-1} \sum_{j=0}^{N} A_j(z,u) \log^{-j} x + R(x,z,u) , \tag{2.21}$$

where $A_j(z,u)$ $(j = 1,\dots,N)$ is an analytic function of z and u for $|z| < \rho$, $|u| < \rho$, continuous for $|z| \le \rho$, $|u| \le \rho$, such that $A_j(z,0) = 0$ and

$$R(x,z,u) = O(x(\log x)^{\operatorname{Re} zu - N - 2}) , \tag{2.22}$$

where the O-constant is uniform in z and u .

We differentiate (2.21) with respect to z, use (2.22) and Cauchy's inequality for the derivative of an analytic function to estimate the error term, and then set $z = 1$, which is possible since $\rho > 1$. This gives uniformly for $|u| \le \rho$

$$\sum_{n\le x} g(n)\, u^{f(n)-1} = x \sum_{j=0}^{N} \log^{u-1-j} x \,(C_j(u) + B_j(u) \log\log x) + O(|u|^{-1} x \log^{\operatorname{Re} u-N-2} x) \tag{2.23}$$

where $C_j(u) = \frac{1}{u} \frac{\partial A_j(z,u)}{\partial z}\Big|_{z=1}$, $B_j(u) = A_j(1,u)$

Note again that $f(n) = 0$ if $n = 1$ and possibly if n is square-full, so that,

$$\sum_{n\le x, f(n)=0} g(n) \ll \sum_{n\le x, f(n)=0} \Omega(n) \ll \log x \sum_{n\le x, n \text{ squarefull}} 1 \ll x^{1/2}\log x ,$$

which implies by (2.23)

$$(2.24)\quad \sum_{n\le x}{}' g(n)\, u^{f(n)-1} = x \sum_{j=0}^{N} \log^{u-1-j} x\,(C_j(u) + B_j(u)\log\log x)$$

$$+ O(x|u|^{-1}\log^{\mathrm{Re}\,u-N-2} x) + O(|u|^{-1}x^{1/2}\log x) .$$

We proceed now as in the proof of Th. 1, integrating (2.24) over u real from $\varepsilon(x) = x^{-2/3}$ to 1, and the proof is very much the same. We note here only that the integral of the left-hand side of (2.24) is

$$\sum_{n\le x}{}' \frac{g(n)}{f(n)} - \sum_{n\le x}{}' \frac{g(n)}{f(n)}(\varepsilon(x))^{f(n)} = \sum_{n\le x}{}' \frac{g(n)}{f(n)} + O(x^{1/3}\log\log x) ,$$

since $g(n) \ll \Omega(n)$, $f(n) \ge 1$ if $f(n) \ne 0$ and $\sum_{n\le x}\Omega(n) \ll x\log\log x$. The integrals of the main terms on the right-hand side of (2.24) are handled exactly the same way as was done in the proof of Th. 1, the integral of the error term is $O(x\log^{-N} x)$, so that after collecting all the terms we obtain the conclusion of the theorem.

3. Applications and some remarks

Theorem 1 can be obviously applied to additive functions $\omega(n)$ and $\Omega(n)$ (and in both cases $\sum'_{n\le x} = \sum_{2\le n\le x}$), since $\omega(p^k) = 1$ and $\Omega(p^k) = k$ for all primes p and integers $k \ge 1$. Therefore (1.1) and (1.2) may be replaced with the sharper asymptotic formula furnished by Th. 1.

The proof of Th. 1 uses essentially the same method presented by De

Koninck in [1], only now instead of a result of A. Selberg [6] Delange's Lemma 1 is used. This lemma is much sharper, but also more restrictive than Selberg's result, so that we had to make a hypothesis (and our condition that $f(p^k) < Ck$ for $k \geq 2$ is easy to verify) which would ensure that R of Lemma 1 is strictly greater than unity, so that (2.4) holds uniformly for $|z| \leq 1$. If f were not only integer-valued, then $z^{f(n)}$ would have a critical point for $z = 0$ and the functions $A_j(z)$ might not be analytic for $z = 0$, which would produce great difficulties in estimating $\int_{\varepsilon(x)}^{1} B_j(z) \log^z x \, dz$, since $\varepsilon(x)$ has to be taken small.

A result like Th. 2 seems to be completely new. From Delange's proof of Lemma 2 it is seen that $F(z) = \prod_p (1 + \sum_{k=2}^{\infty} g_z(p^k) p^{-k})$, where the function $g_z(n) = \sum_{d|n} \mu(d) z^{f(n/d)}$ is the Möbius inverse of $z^{f(n)}$. Taking in particular $f(n) = \Omega(n) - \omega(n)$, it is seen that this function satisfies the hypothesis of Th. 2 and that

$$F(z) = \prod_p (1 - 1/p)(1 + 1/(p-z)) = \sum_{k=0}^{\infty} d_k z^k , \tag{3.1}$$

where it is well-known (see [3] and [5]) that

$$d_k = \lim_{x\to\infty} x^{-1} \sum_{n\leq x, \Omega(n)-\omega(n)=k} 1 , \tag{3.2}$$

which means that d_k is the density of integers n for which $\Omega(n) - \omega(n) = k$. Therefore in the case when $f(n) = \Omega(n) - \omega(n)$ one obtains (2.2) (noting that $d_0 = 1/\zeta(2) = 6/\pi^2$) with

$$e_0 = \int_0^1 (F(z) - 6/\pi^2) z^{-1} dz = \int_0^1 \sum_{k=1}^{\infty} d_k z^{k-1} dz = \sum_{k=1}^{\infty} d_k/k ,$$

and all the other e_i's are also computable.

One could also generalize Th. 2 by supposing that $f(p) = \ldots = f(p^{r-1}) = 0$, $f(p^r) = 1$ and $0 < f(p^k) < Ck$ for $k \geq r + 1$, where $r \geq 2$ is a fixed natural number. In that case we could find an estimate for $\sum_{n\leq x} z^{f(n)}$ (using the methods of Delange [4]) which would lead to the formula (2.2) with $x^{1/r}$ instead of $x^{1/2}$.

Also it may be observed that Theorem 3 may be applied to $g(n) = \Omega(n)$, $f(n) = \omega(n)$, thus improving the asymptotic formula for $\sum'_{n\leq x} \Omega(n)/\omega(n)$ proved by De Koninck [2].

Finally we wish to thank Prof. H. Delange for his suggestions and criticism of an earlier version of this paper.

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* Research financed by the Mathematical Institute of Belgrade and Republička Zaj. of Serbia

(Received July 10, 1979)