Journal of the Indian Math, Soc.
42 (1978) 353-365

SUMS TAKEN OVER $n \leqslant x$ WITH PRIME FACTORS $\leqslant y$ OF $z^{\Omega(n)}$, AND THEIR DERIVATIVES WITH RESPECT TO z

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[Received April 17, 1976; Revised August 18, 1978]

0. Introduction. The problems discussed here have their roots in two seemingly remote questions. The first problem is to estimate $\sum_{n \leqslant x} z^{\Omega(n)}$, where z is a real or complex variable and where $\Omega(n)$ denotes the total number of prime factors of n. The most successful, perhaps, of treatments of this problem before 1960 was by Selberg **[10]**. Other writers included Bateman **[1]**, Grosswald **[6]**, and Sathe.

The other problem was to estimate $\Psi(x, y)$, the number of integers $\leqslant x$ all of whose prime factors are $\leqslant y$. In a series of papers during the 50's [2, 4, 3] N. G. de Bruijn gave an estimate of $\Psi(x, y)$ which was uniform in x and y and useful so long as $\dfrac{\log x}{\log y}$ was not "too large."

Here we propose to estimate $\Sigma\, z^{\Omega(n)}$, the sum to be taken over $\{n : n \leqslant x,\ c < p \leqslant y \text{ if } p \mid n, p \text{ prime}\}$. (Hereafter the words "the sum (or product) to be taken over" will be omitted; the set of summation will follow the expression. Also p will denote a prime.) This sum is equal to $\Sigma\, z^{\Omega(n)}\, \{n : n \leqslant x\}$ if $c < 2$ and $y \geqslant x$, and to $\Psi(x, y)$ if $c < 2$, $z = 1$. Let $\Psi(x, y, z, c) = \Sigma\, z^{\Omega(n)}\, \{n : n \leqslant x,\ c < p \leqslant y \text{ if } p \mid n\}$. Our goal is to give an approximation to Ψ which includes both the above mentioned problems as special cases, is uniform in x, y, and z, such that the approximation is differentiable with respect to y and to z, and such that the n-th derivative ∂z of the approximating function estimates $\left(\dfrac{\partial^n}{\partial z^n}\Psi\right)$. With some qualifications, this can be done. In §1 we apply results of [5] and [10] to the case $y = x$, in which there is no restriction on the prime factors of n. In § 2 our methods follow those of [4].

§1. **The One Parameter Problem.** Here our goal is to estimate $\Psi(x, x, z, c) = \Sigma\, z^{\Omega(n)}\, \{n : n \leqslant x,\ c < p \text{ if } p \mid n\}$. Selberg [10] showed explicitly that if $c < 2$, $|z| < 2$ then as $x \to \infty$,

$$\sum_{n \leqslant x} z^{\Omega(n)} = xH(z)(\log x)^{z-1} + O\left(x(\log x)^{Re z-2}\right),$$

where

$$H(z) = \frac{1}{\Gamma(z)} \Pi \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{z},$$

the product to be taken over all primes p.

If $|z| > c$ in general the sum does not increase smoothly with x, but makes big jumps when x is a power of some prime p, $c < p \leqslant |z|$. We do not attempt to give estimates in this case. For $c > |z|$, $c \geqslant 2$ one has, following Selberg, $\Psi(x, x, z, c) = \Sigma\, z^{\Omega(n)}\, \{n : n \leqslant x; c < p \text{ if } p \mid n\} = xG(z)(\log x)^{z-1} + O\left(x \log x\right)^{Re z-2})$,

where

$$G(z) = \frac{1}{\Gamma(z)} \underset{\{p : p \leqslant c\}}{\Pi} \left(1 - \frac{1}{p}\right)^{z} \underset{\{p : p > c\}}{\Pi} \left(1 - \frac{1}{p}\right)^{z} \left(1 - \frac{z}{p}\right)^{-1}.$$

Delange [5] extends these formulas as follows. For any positive c, any z with $|z| < c$, and any integer $q \geqslant 1$, as $x \to \infty$ there exist $A_j(z)$ not depending on q, holomorphic in $|z| < c$ such that

$$\Psi(x, x, z, c) = x(\log x)^{z-1}\left[H(x, z) + O\left(\frac{1}{(\log x)^{q+1}}\right)\right] \tag{1}$$

where

$$H(x, z) = \sum_{j=0}^{q} A_j(z)(\log x)^{-j}.$$

If p' is the least prime $> c$, and $\varepsilon > 0$, the $A_j(z)$ converge for $|z| \leqslant p' - \varepsilon$ and the above "O" is uniform for $|z| \leqslant p' - \varepsilon$.

Now let

$$\Lambda_n(x, y, z, c) = \frac{\partial^n}{\partial z^n} \Psi(x, y, z, c) \tag{2}$$

Returning to $\Psi(x, x, z, c)$, we write (1) as

$$\Psi(x, x, z, c) = x(\log x)^{z-1} \sum_{j=0}^{q} A_j(z)(\log x)^{-j} + B(z)\, x(\log x)^{z-q-2} \tag{3}$$

with $B(z) = 0(1)$ uniformly for $|z| < p' - \varepsilon$. Let us say $|B(z)| < C_1$ for all x sufficiently large and for all z, $|z| < p' - \varepsilon$. Now assume that x is large enough and write $g(x, z) = B(z)(\log x)^z$ so that

$$\frac{\partial}{\partial z} g(x, z) = B'(z)(\log x)^z + B(z)(\log x)^z \log\log x.$$

Now for each z such that

$$|z| < p' - \varepsilon, \text{ let } r(z) = p' - \varepsilon - |z| > 0.$$

By Cauchy's inequality, we have for $|z| < p' - \varepsilon$

$$|B'(z)| \leqslant C_1/r(z).$$

Thus for $|z| < p' - 2\varepsilon$ one has uniformly $|B'(z)| \leqslant C_1/\varepsilon$, and in this disc $\partial/\partial z\, g(x, z) = O_\varepsilon(\log^z x \log\log x)$. Therefore differentiating (3) with respect to z and using (2), we obtain

$$\begin{aligned}\Lambda_1(x, x, z, c) &= x \log^{z-1} x \log\log x \cdot \sum_{j=0}^{q} A_j(z) \log^{-j} x + \\ &\quad + x \log^{z-1} x \sum_{j=0}^{q} A_j'(z) \log^{-j} x + O(x \log^{z-q-2} x \log\log x) \\ &= x \log^{z-1} x \sum_{j=0}^{q} [A_j(z) \log\log x - A_j'(z)] \log^{-j} x + \\ &\qquad + O(x \log^{z-q-2} x \log\log x). \end{aligned} \tag{4}$$

Further differentiation is now possible since both $|B(z)|$ and $|B'(z)|$ are bounded in $|z| < p' - 2\varepsilon$. At each differentiation the disc shrinks in radius by ε, and the bound increases by a factor of $1/\varepsilon$.

In this way we have eventually the following: For fixed integer n, $q \geqslant 1$, p' prime number, and real $c < p'$, let $\varepsilon > 0$ satisfy $n\varepsilon < \frac{1}{2}(c - p')$. Let

$$f(x, z) = x \log^{z-1} x \sum_{j=0}^{q} A_j(z) \log^{-j} x. \tag{5}$$

Then

$$\Lambda_n(x, x, z, c) = \frac{\partial^n}{\partial z^n} f(x, z) + O_\varepsilon(x \log^{z-q-2} x (\log\log x)^n), \tag{6}$$

with the "O" uniform in z for $|z| \leqslant c$.

REMARK. Normally one cannot differentiate "O", but in this case the "O" is with respect to x, while the differentiation is with respect to z. Further, the functions involved are holomorphic in z. Although we shall not pursue this in such detail as the derivatives, one can also integrate $f(x, z)\,dz$ and the result will approximate $\sum \frac{z^{\Omega(n)}}{\Omega(n)}$ $\{n : n \leqslant x$ and $p > c$ if $p \mid n\}$.

In fact, let us take $c > 1$ fixed, $0 \leqslant z \leqslant c$ real, $q \geqslant 1$ an integer, and $\varepsilon > 0$. Let $F(x, z) = \Psi(x, x, z, c)$ and let $f(x, z)$ be as in (5). Now

$$F(x, z) = f(x, z) + O(x \log^{z-q-2} x), \tag{7}$$

uniformly in $0 \leqslant z \leqslant c - \varepsilon$. Thus

$$\int_0^z F(x, t)\,dt = \int_0^z f(x, t)\,dt + O\,(x \log^{z-q-2} x)(\log\log x)^{-1}$$

uniformly in $0 \leqslant z \leqslant c - \varepsilon$. In other words,

$$\sum \frac{z^{1+\Omega(n)}}{1+\Omega(n)} \{n : n \leqslant x,\ c < p \text{ if } p \mid n\}.$$

$$= \int_0^z f(x, t)\,dt + O\,(x \log^{z-q-2} x)\,(\log\log x)^{-1}$$

uniformly for $0 \leqslant z \leqslant c - \varepsilon$.

Though it is more difficult, if $1 < z \in \mathbf{R}$ one can integrate $F(x, t)$ with respect to t from $z - 1$ to z and obtain an estimate for

$$\sum \frac{z^{\Omega(n)}}{\Omega(n)} \{n : 2 \leqslant n \leqslant x,\ c < p \text{ if } p \mid n\}.$$

The result is that

$$\sum \frac{z^{\Omega(n)}}{\Omega(n)} \sim \frac{B_1(z)\, x \log^{z-1} x}{\log\log x},$$

where $B_1(z) = z^{-1} A_0(z)$. We shall not need this result in what follows, and the proof is omitted.

Although it is of some intrinsic interest to work out the applications of known general results to $\Psi(x, x, z, c)$ and its derivatives and integral, we have another purpose for these estimates.

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Although it is of some intrinsic interest to work out the applications of known general results to $\Psi(x, x, z, c)$ and its derivatives and integral, we have another purpose for these estimates.

With $y < x$, $\Psi(x, y, z, c)$ reflects the multiplicative structure of the integers $\leqslant x$ through two effects: First, $\Omega(n)$ depends on the prime factorization of n, and second, the sum is taken only over numbers with small prime factors ($p \leqslant y$ if $p \mid n$). We shall develop $\Psi(x, y, z, c)$ estimates by a recursion in which y shrinks away from x. Thus, as in any induction, a knowledge of the initial conditions ($y = x$) for the recursion is necessary. This precondition has been filled in the preceding pages, and we move to the two parameter problem, in which both x and y vary.

§2. The Two Parameter Problem. We now turn to estimating $\Psi(x, y, z, c)$ in the case where $y < x$. Since $\log x/\log y$ appears frequently, let $u = \log x/\log y$.

There now appear several definitions and elementary propositions which we shall need in what follows. A few have been repeated for easier reading.

DEFINITION:

$$\Psi(x, y, z, c) = \sum z^{\Omega(n)}$$
$$\{n: n \leqslant x,\ c < p \leqslant y \text{ if } p \mid n\} \text{ for } x, y \geqslant 1,\ |z| < c. \tag{8}$$

DEFINITION:

$$\Lambda_r(x, y, z, c) = \sum z^{\Omega(n)-r} \prod (\Omega(n) - m), \tag{9}$$

where the product runs through $\{m: 0 \leqslant m < r\}$ and the sum is taken over $\{n: n \leqslant x$ and $c < p \leqslant y$ if $p \mid n\}$ for $r \geqslant 0$ integer, $x \geqslant 1$, $y \geqslant 1$, $|z| < c$.

Let $\Lambda_{-1}(x, y, z, c)$ be identically 0. (This will simplify the statement of some of our proofs and results.) Thus

$$\Lambda_0(x, y, z, c) = \Psi(x, y, z, c) \quad \text{and}$$
$$\Lambda_r(x, y, z, c) = (\partial^r/\partial z^r)(\Psi(x, y, z, c)). \tag{10}$$

DEFINITION:

$$F_r(x, z) = \Lambda_r(x, x, z, c) \tag{11}$$

If $r = 0$ we omit r.

DEFINITION.

$$\psi(x, y, z, c) = \Psi(x, y, z, c) \text{ for } y \geqslant x \geqslant 1, \text{ and}$$

$$\frac{\partial}{\partial y}\psi(x, y, z, c) = (z/\log y)\,\psi(x/y, y, z, c) \text{ for } 1 \leqslant y \leqslant x. \quad (12)$$

DEFINITION.

$$\lambda_r(x, y, z, c) = \Lambda_r(x, y, z, c) \text{ for } y \geqslant x \geqslant 1, \text{ and}$$

$$\frac{\partial}{\partial y}\lambda_r(x, y, z, c) = (z/\log y)\,\lambda_r(x/y, y, z, c) + (r/\log y)\,\lambda_{r-1}(x/y, y, z, c) \text{ for } 1 \leqslant y < x, r \geqslant 0. \quad (13)$$

DEFINITION.

$$f_z(u) = 0 \text{ for } u < 0,\ 1 \text{ for } 0 \leqslant u \leqslant 1, \text{ and}$$

$$-uf_z'(u) = zf_z(u-1) \text{ for } u > 1. \quad (14)$$

DEFINITION.

$$f_{z,r}(u) = 0 \text{ for } u < 0,\ 1 \text{ for } 0 \leqslant u \leqslant 1, \text{ and}$$

$$-uf_{z,r}'(u) = zf_{z,r}(u-1) + rf_{z,r-1}(u-1) \text{ for } u > 1. \quad (15)$$

DEFINITION.

$$\rho_z(u) = 0 \text{ for } u < 0,\ G(z) \text{ for } 0 \leqslant u \leqslant 1, \text{ and}$$

$$-u^z\rho_z'(u) = z(u-1)^{z-1}\rho_z(u-1) \text{ for } u > 1, \quad (16)$$

where $G(z) = (\Gamma(z))^{-1}\Pi(1-p^{-1})^z\ \Pi(1-p^{-1})^z\ (1-zp^{-1})^{-1}\{p: p \leqslant c\}$ $\{p: p > c\}$.

DEFINITION. $R(y)$ denotes an arbitrary but fixed function of y such that $R(y) \downarrow 0$, $R(y) > (\log y)/y$ for $y \geqslant 2$, and $|\pi(y) - \mathrm{Li}(y)| < yR(y)/\log y$ $(y \geqslant 2)$. One could e.g. take $R(y) = C\exp(-C\sqrt{\log y})$. (17)

REMARK. We follow the common practice of letting each instance of "C" stand for a possibly different constant.

The following propositions set the stage for the main result. We shall now fix c, and without loss of generality take c to be of the form $p' - \varepsilon$ fixed. Consequently we omit c as a variable in Ψ, Λ, ψ, λ etc.

PROPOSITION 1. $\Psi(x, y^h, z) - \Psi(x, y, z) = z\Sigma\Psi(x/p, p, z)\{p: y < p \leqslant y^h\}$ *for* $h > 1$ *real.*

PROOF. Immediate from the definition of Ψ.

PROPOSITION 2. $\Lambda_r(x, y^h, z) - \Lambda_r(x, y, z) = \Sigma(z\Lambda_r(x/p, p, z) + r\Lambda_{r-1}(x/p, p, z))\{p: y < p \leqslant y^h\}$ *for* $h \geqslant 1$ real, $r \geqslant 0$ *integer.*

PROOF. Also from the definition.

PROPOSITION 3. $\psi(x, y^h, z) - \psi(x, y, z) = z\int_y^{y^h}\psi(x/t, t, z)\,dt/\log t$ *for* $h \geqslant 1$.

PROPOSITION 4. $\lambda_r(x, y^h, z) - \lambda_r(x, y, z) =$

$$= \int_y^{y^h}(z\lambda_r(x/t, t, z) + r\lambda_{r-1}(x/t, t, z))\,dt/\log t \textit{ for } h \geqslant 1.$$

Proofs of 3 and 4. From the definition.

PROPOSITION 5. $\psi(x, y, z) = x\int_0^\infty f_z\left(\frac{\log x - \log t}{\log y}\right)d\left(\frac{F(t, z)}{t}\right)$

(Here t is the variable of integration.)

PROPOSITION 6. $x\log^{z-1}x\rho_z(u) =$

$$= x\int_1^\infty f_z\left(\frac{\log x - \log t}{\log y}\right)d\left(\frac{G(z)\,t\log^{z-1}t}{t}\right)$$

where again t is the variable of integration.

Proofs are both immediate from the respective definitions.

REMARK. Our intent is emerging. It is to estimate Ψ by ψ and ψ by $x\log^{z-1}x\rho_z(u)$. This last estimate involves no summations or other number theoretic considerations. For the most part we omit the second step, from ψ to $x\log^{z-1}x\rho_z(u)$. Proposition 6 can be generalized to contain a finite series of decreasing powers of $\log x$, and a similar result holds for λ_r as well as ψ. Note that the difference between $\psi(x, y, z)$ and $x\log^{z-1}x\rho_z(u)$ is "small" because $F(t, z)$ is close to $G(z)\,t\log^{z-1}t$. (See Introduction, (1).)

PROPOSITION 7. *If $f_z(u)$ has finite L_2 norm, then $\hat{f}_z(t)=0$ if $t \leqslant 0$, and*

$$\hat{f}_z(t) = C \exp \int_1^t \frac{1}{s}(ze^{-is}-1)\,ds \text{ if } t>0. \tag{18}$$

PROOF. Formal calculations give

$$\hat{f}_z(t) = C \exp \int_1^t \frac{1}{s}(ze^{-is}-1)\,ds \tag{19}$$

Thus equality holds if and only if (19) has finite L_2 norm. This occurs if and only if $z=1$ or $Re(z)>1$. Note also that if $Re(z)>1$, $\hat{f}_z(0)=0$ so that $\int_0^\infty f_z(u)\,du=0$. This also follows from integration by parts. For there exists a sequence b_n of real numbers such that $b_n>n$ and such that $b_n f_z(b_n)=O(1)$. (Because f_z has finite L_2 norm.) And if one integrates by parts the expression $\int_0^{b_n} f_z(u)\,du$, in view of (14) the limit as $n\to\infty$ is 0.

§3. **The main results.** We shall be principally concerned with proving that Ψ is close to ψ. As shown by propositions of §2, ψ is close to $x\log^{z-1}x\rho_z(u)$. This last can be improved by replacing $G(z)t\log^{z-1}t$ in Proposition 6 by the series estimate (1) of §1. These functions in turn are accessible to study via Fourier series techniques, as indicated by Proposition 7.

THEOREM 1. *For all integers $r\geqslant 0$, for all p' prime, for all $\varepsilon>0$ there exists $C>0$ such that for all $x>3$, for all $y>2$, for all z such that $|z|<c=p'-\varepsilon$, we have*

$$|\Lambda_r(x,y,z,c)-\lambda_r(x,y,z,c)|<$$
$$<|Cx\log^{z-1}x(\log\log x)^r u^{|z|}(1+\log(u+1))R(y)|. \tag{20}$$

REMARK. In less precise language, $\Lambda_r(x,y,z,c)$ is near $\lambda_r(x,y,z,c)$ uniformly in x, y, and z throughout the indicated range. Recall that $u=\log x/\log y$ so that the estimate is of no value if, for instance, $u>x$.

COROLLARY 1. *With the same conditions as above, when $r=0$ we have*

$$|\Psi(x,y,z,c)-\psi(x,y,z,c)|<$$
$$<|Cx(\log x)^{z-1}u^{|z|}(1+\log(u+1)R(y)|. \tag{21}$$

COROLLARY 2. *With $z=1$, $r=0$, $c<2$ one has an estimate for $\Psi(x,y)$, the number of integers $\leqslant x$ all of whose prime factors are $\leqslant y$.*

$$|\Psi(x,y)-\psi_1(x,y,1,1)|<Cxu(\log u+1)R(y). \tag{22}$$

REMARK. Corollary 2 improves slightly on **[4]**, where one had u^2 in place of $u(\log u+1)$.

PROOF OF THEOREM 1. Fix $r\geqslant 0$, p' prime, $\varepsilon>0$, and $c=p'-\varepsilon$. Again write Ψ, ψ, Λ, λ without c. Let

$$\Delta=\Delta(x,y,h,z,r)=\Sigma z\Lambda_r(x/p,p,z)+$$
$$+r\Lambda_{r-1}(x/p,p,z)\{p: y<p<y^h\}-$$
$$-\int_y^{y^h}(z\Lambda_r(x/t,t,z)+r\Lambda_{r-1}(x/t,t,z))\,dt/\log t. \tag{23}$$

Thus Δ is the error that results from replacing $d\pi(t)$ by $dt/\log t$. There is an heuristic principle that this error is small, and the usual proof is an exercise in the application of the prime number theorem.

Let $\Delta(n)=\Delta(x,y,h,z,n)=\theta(n,z)\left(\sum_{p\in J}1-\int_J dt/\log t\right)$ where J is the interval $[p(n),x/n]\cap[y,y^h]$, $p[n]$ denotes the largest prime factor of n, and where $\theta(n,z)=0$ if some prime factor of n is $\leqslant c$, else

$$\theta(n,z)=z^{\Omega(n)-r+1}\prod_{m=-1}^{r-2}(\Omega(n)-m).$$

Note that $\sum_{n=1}^\infty \Delta(n)=\Delta$, a direct consequence of the definitions of Δ and of $\Delta(n)$. The sum is finite since if $n>x/y$, J is empty and $\Delta(n)=0$.

We separate the range $1\leqslant n\leqslant x/y$ in which $\Delta(n)$ may be non-zero into two intervals: $1\leqslant n\leqslant x/y^h$ and $x/y^h<n\leqslant x/y$. If $n\leqslant x/y^h$,

$|\Delta(n)| \leqslant C\theta(n, z)\, y^h R(y)/\log y$. And if $x/y^h < n \leqslant x/y$, $|\Delta(n)| \leqslant C\theta(n, z)\, xR(y)/n \log y$.

LEMMA 1. *There exists $C > 0$ such that for all z satisfying $|z| < p' - \varepsilon$, for all $x \geqslant 3$, $y \geqslant 2$, for all h satisfying $1 \leqslant h \leqslant 2$,*

$$|\Delta(x, y, h, z, r)| \leqslant |Cx \log^{z-1} x (\log\log x)^r R(y)\cdot(h - 1 + 1/\log y)|. \tag{24}$$

PROOF OF LEMMA 1. Summing the estimates of $\Delta(n)$, we have

$$\sum_{n \leqslant x/y^h} Cy^h R(y)\theta(n, z)/\log y \leqslant (Cy^h R(y)/\log y) \sum_{n \leqslant x/y^h} \theta(n, z). \tag{25}$$

Now $\theta(n, z) = z \cdot z^{\Omega(n) - r} \prod_{m=0}^{r-1} (\Omega(n) - m) + rz^{\Omega(n) - r + 1} \prod_{m=0}^{r-2} (\Omega(n) - m)$, so $\sum_{n \leqslant x/y^h} \theta(n, z) = z\Lambda_r(x/y^h, x/y^h, z) + r\Lambda_{r-1}(x/y^h, x/y^h, z)$. Now from (6) of §1 we have $\Lambda_r(s, s, z) \leqslant Cs \max(1, |\log^{z-1} s|) \max(1, \log\log s)^r)$. Thus this last sum of two terms is $\leqslant C(x/y^h) \max(1, |\log^{z-1}(x/y^h)|) \max(1, (\log\log(x/y^h))^r)$, and

$$\sum_{n \leqslant x/y^h} |\Delta(n)| \leqslant |C x R(y) \log^{z-1} x (\log\log x)^r/\log y|. \tag{26}$$

To prove the lemma we need a similar estimate for the range $x/y^h \leqslant n \leqslant n/y$. We proceed to give one.

We have

$$\sum_{x/y^h < n \leqslant x/y} C x R(y)\theta(n, z)/n \log y = C x R(y)/\log y) \sum_{x/y^h < n \leqslant x/y} \theta(n, z)/n.$$

And

$$\begin{aligned}\sum_{x/y^h < n \leqslant x/y} \theta(n, z)/n &= \int_{x/y^h}^{x/y} (z/t)\, d\Lambda_r(t, t, z) + \\ &+ \int_{x/y^h}^{x/y} (r/t)\, d\Lambda_{r-1}(t, t, z) \quad [\text{variable of integration } t] \\ &= (z/t)\Lambda_r(t, t, z)\Big|_{t=x/y^h}^{t=x/y} + (r/t)\Lambda_{r-1}(t, t, z)\Big|_{t=x/y^h}^{x/y} + \\ &+ \int_{x/y^h}^{x/y} (z/t^2)\Lambda_r(t, t, z)\, dt + \int_{x/y^h}^{x/y} (r/t^2)\Lambda_{r-1}(t, t, z)\, dt\end{aligned} \tag{27}$$

Now from (6) with $q = 0$ we have, since $A_0(z)$ and its first r derivatives are bounded for $|z| \leqslant C$,

$$|\Lambda_r(t, t, z)| \leqslant Ct\, |\log^{z-1} t|\, (\log\log t)^r \tag{28}$$

for $|z| \leqslant c$ and $t \geqslant 3$. For $t < 3$ we note simply that $|\Lambda_r(t, t, z)| \leqslant C$.

Now in view of (28), the first two terms of (27) are $\leqslant C|\log^{z-1} x|(\log\log x)^r$, and the integrals of (27) are $\leqslant C|\log^{z-1} x|(\log\log x)^r$. $\int_{x/y^h}^{x/y} dt/t = C|\log^{z-1} x|(\log\log x)^r (h - 1)\log y$. Together with (26), this proves Lemma 1.

Recalling that c and r are fixed, we define $D_r(x, y, z) = \Lambda_r(x, y, z) - \lambda_r(x, y, z)$. We must show that D_r is small. From Propositions 2 and 4, with the definition of Δ, D_r satisfies the approximate recursion

$$\begin{aligned}D_r(x, y^h, z) - D_r(x, y, z) &= \int_y^{y^h} z D_r(x/t, t, z)\, dt/\log t + \\ &+ \int_y^{y^h} r D_{r-1}(x/t, t, z)\, dt/\log t + \Delta(x, y, h, z, r).\end{aligned} \tag{29}$$

Thus $|D_r(x, y, z)| \leqslant |D_r(x, y^h, z)| + |z| \int_y^{y^h} |D_r(x/s, s, z)|\, ds/\log s + r\int_y^{y^h} |D_{r-1}(x/s, s, z)|\, ds/\log s + |\Delta(x, y, h, z, r)|$.

Note that $D_r(x, x) = 0$. If we were to start with $y = x$ and decrease y, the recursion would keep $D_r(x, y) = 0$ were it not for Δ. Despite Δ the error is controllable. We now prove Theorem 1.

Suppose that for some integer $l > 0$,

$$|D_{r'}(x, y, z)| \leqslant |C(l)\, x \log^{z-1} x\, R(y)\cdot(\log\log x)^{r'} \tag{30}$$

if $y^l \geqslant x$ and $r' \leqslant r$.

For y such that $y^{l+1} > x \geqslant y^l$ let $h = 1 + \frac{1}{l}$. Now

$$| D_{r'}(x, y, z) | \leqslant D_{r'}(x, y^h, z) | + z \int_y^{y^h} | D_{r'}(x/s, s, z) | \, ds/\log s +$$
$$+ r' \int_y^{y^h} | D_{r'-1}(x/s, s, z) | \, ds/\log s + | \Delta (x, y, h, z, r') |$$

Using our induction assumption, this is

$$\leqslant C(l) \, | x \log^{z-1} x \, R(y) (\log \log x)^{r'} | +$$
$$+ \left| z \int_y^{y^h} C(l) \frac{x}{s} \log^{z-1}(x/s) \, R(y) (\log \log x)^{r'} ds/\log s \right| +$$
$$+ r' \left| \int_y^{y^h} C(l) \frac{x}{s} \log^{z-1}(x/s) \, R(y) (\log \log x)^{r'-1} ds/\log s \right| +$$
$$+ | C x \log^{z-1} x (\log \log x)^{r'} | .$$

Evaluating and simplifying, this is

$$\leqslant C(l) \, | x \log^{z-1} x \, R(y) (\log \log x)^{r'} (1 + | z | / l + r'/l \log \log x) | +$$
$$+ C x \, | \log^{z-1} x (\log \log x)^{r'} | .$$

Let $C(l+1) = C(l) (1 + | z | / l + r'/l \log \log x) + C$. Then the induction hypothesis (27) is satisfied for $l + 1$.

It remains to bound $C(l)$ as $l \to \infty$. From the recursion for $C(l)$ we have $C(l) = o\,(l^{|z| + r/\log \log x} \log l)$. Since $u = \log x / \log y$, if $y^l \geqslant x$, $u \leqslant l$. Replacing l with u, then, gives Theorem 1. (The factor $u^{r/\log \log x}$ which appears in the final estimate is in fact bounded, since $u = \log x / \log y$ and $x \geqslant y$, and so is not included in the statement of Theorem 1.)

§ **4. Applications.** If z is a positive integer, $\int \frac{1}{x} \, d\Psi \, (x, y, z)$ (x the variable) is important to the estimation of

$$\Sigma \, \mu^2(n) \prod_{p|n} (1 - z/p) \, \{n \colon c < p \leqslant y \text{ if } p \mid n, n \leqslant x\}.$$

This expression occurs in large sieve estimates when a set of x consecutive integers is sieved by removing z congruence classes mod p for each prime p, $c < p \leqslant y$. For more on this, see **[7, 8]**. Another reason for requiring $| z | \leqslant c$ is apparent here; if $z = p$, the set is cleaned out entirely when all the congruence classes are removed mod p.

To estimate $\Sigma \, \Omega(n) \, \{n \leqslant x, p \leqslant y \text{ if } p \mid n\}$ let $z = r = 1$. To estimate sums of polynomials in $\Omega(n)$ taken over the same set as above use linear combinations of $\Lambda_r(x, y, 1)$ for $0 \leqslant r \leqslant \deg$ (polynomial).

To estimate $\Sigma \, 1 \, \{n \colon \Omega(n) \equiv W \mod q, n \leqslant x, p \leqslant y \text{ if } p \mid n\}$, take a linear combination of $\Psi(x, y, \alpha)$ over the qth roots α of 1. Thus e.g. with $\alpha = \exp(2\pi i/3)$, $\Sigma \, 1 \, \{n \colon n \leqslant x, \; \Omega(n) \equiv 1 \mod 3, \; p \leqslant y \text{ if } p \mid n\} = \frac{1}{3} (\alpha^2 \Psi(x, y, \alpha) + \alpha \Psi(x, y, \alpha^2) + 1 \, \Psi(x, y, 1))$. We hope that this does not exhaust the list of applications.

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