

AN ASYMPTOTIC FORMULA FOR RECIPROCAL OF LOGARITHMS OF CERTAIN MULTIPLICATIVE FUNCTIONS

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Sums of the form $\sum'_{n \leq x} 1/\log f(n)$, where $f(n)$ is a multiplicative arithmetical function and \sum' denotes summation over those values of n for which $f(n) > 0$ and $f(n) \neq 1$, were studied by De Koninck [2], De Koninck and Galambos [3], Brinitzer [1] and Ivić [5]. The aim of this note is to give an asymptotic formula for $\sum'_{n \leq x} 1/\log f(n)$ for a certain class of multiplicative, positive, prime-independent functions (an arithmetical function is prime-independent if $f(p^\nu) = g(\nu)$ for all primes p and $\nu = 1, 2, \dots$). This class of functions includes, among others, the functions $a(n)$ and $\tau^{(e)}(n)$, which represent the number of non-isomorphic abelian groups of order n and the number of exponential divisors of n respectively, and none of the estimates of the above-mentioned papers may be applied to this class of functions. We prove the following.

THEOREM. *Let $f(n)$ be a multiplicative arithmetical function such that for all primes p and $\nu = 1, 2, \dots$ we have $f(p^\nu) = g(\nu)$, where $g(1) = 1$, $g(\nu) > 1$ for $\nu \geq 2$ and $\liminf_{\nu \rightarrow \infty} g(\nu) > 1$. Then we have*

$$(1) \quad \sum'_{n \leq x} 1/\log f(n) = x \int_{-\infty}^0 (C(t) - 6/\pi^2) dt + O(x^{1/2} \log^{1/2} x),$$

where $C(t) = \prod_p (1 + \sum_{k=2}^{\infty} (g^t(k) - g^t(k-1))p^{-k})$, and \sum' denotes summation over those values of n for which $f(n) > 1$.

Proof. First of all $f(n) \geq 1$, and $f(n) = 1$ if and only if n is square-free, or equivalently if and only if $1 - \mu^2(n) = 0$, where $\mu(n)$ is the Möbius function.

Let us define

$$(2) \quad \sum'_{n \leq x} f'(n) = \sum_{n \leq x, f(n) > 1} f'(n).$$

Then we have

$$(3) \quad \begin{aligned} \sum'_{n \leq x} f'(n) &= \sum_{n \leq x} (1 - \mu^2(n))f'(n) = \sum_{n \leq x} f'(n) - \sum_{n \leq x} \mu^2(n) \\ &= \sum_{n \leq x} f'(n) - \frac{6}{\pi^2} x + O(x^{1/2} \exp(-C \log^{3/5} x (\log \log x)^{-1/5})), \end{aligned}$$

where C is a positive constant (see [9]).

Received by the editors September 19, 1977 and, in revised form, February 6, 1978.

We now proceed to estimate $\sum_{n \leq x} f^t(n)$ for $t \leq 0$. For $\text{Re } s > 1$ we clearly have

$$\begin{aligned} \sum_{n=1}^{\infty} f^t(n)n^{-s} &= \prod_p (1 + p^{-s} + g^t(2)p^{-2s} + g^t(3)p^{-3s} + \dots) \\ &= \zeta(s) \prod_p \left(1 + \sum_{k=2}^{\infty} (g^t(k) - g^t(k-1))p^{-ks} \right) = \zeta(s)g(s, t), \end{aligned}$$

where $g(s, t) = \sum_{n=1}^{\infty} h(n, t)n^{-s}$, $h(n)$ is multiplicative and

$$h(p^j, t) = \begin{cases} 0 & j = 1, \\ g^t(j) - g^t(j-1) & j \geq 2. \end{cases}$$

Since $t \leq 0$ we have $|h(n, t)| \leq u(n)$, where

$$u(n) = \begin{cases} 0 & \text{if there is a } p \text{ such that } p \parallel n, \\ 1 & \text{otherwise,} \end{cases}$$

so that $\sum_{n \leq x} |h(n, t)| \leq \sum_{n \leq x} u(n) = O(x^{1/2})$. Denoting $w = g^t(2)$ for shortness, further factoring yields

$$(4) \quad g(s, t) = \zeta^{w-1}(2s)u(s, t),$$

where for $t \leq 0$,

$$\begin{aligned} u(s, t) &= \prod_p (1 - p^{-2s})^{w-1} (1 + (w-1)p^{-2s} + (g^t(3) - w)p^{-3s} + \dots) \\ &= \prod_p (1 + (g^t(3) - w)(1 - p^{-2s})^{w-1} p^{-3s} + O(p^{-4s})), \end{aligned}$$

so that if $\sum_{n=1}^{\infty} c(n, t)n^{-s} = u(s, t)$, then for every $\epsilon > 0$ and uniformly in $t \leq 0$,

$$\sum_{n \leq x} |c(n, t)| = O(x^{1/3+\epsilon}),$$

and partial summation gives

$$(5) \quad \sum_{n > x} |c(n, t)| n^{-1/2} = O(x^{-1/6+\epsilon}).$$

If we set $\sum_{n=1}^{\infty} b(n, t)n^{-s} = \zeta^{w-1}(2s)$, then we have by a result of A. Selberg [7]

$$\sum_{n \leq x} b(n, t) = \Gamma^{-1}(w-1)2^{2-w}x^{1/2} \log^{w-2} x + O(x^{1/2} \log^{w-3} x),$$

which gives uniformly in $t \leq 0$

$$(6) \quad \sum_{n \leq x} b(n, t) = O(x^{1/2} \log^{w-2} 2x).$$

From (4) it follows that

$$\sum_{n \leq x} h(n, t) = \sum_{n \leq x} c(n, t) \sum_{m \leq x/n} b(m, t) = O(x^{1/2} \sum_{n \leq x} |c(n, t)| n^{-1/2} \log^{w-2} 2x/n).$$

Since $\sum_{n=1}^{\infty} |c(n, t)| n^{-1/2}$ converges we have

$$\begin{aligned} \sum_{n \leq x} |c(n, t)| n^{-1/2} \log^{w-2} 2x/n &= \sum_{n \leq x^{1/2}} + \sum_{x^{1/2} < n \leq x} \\ &= O(\log^{w-2} x) + O\left(\sum_{n > x^{1/2}} |c(n, t)| n^{-1/2}\right) = O(\log^{w-2} x) + O(x^{-1/12+\epsilon}) = O(\log^{w-2} x) \end{aligned}$$

so that

$$\sum_{n \leq x} h(n, t) = O(x^{1/2} \log^{w-2} x) = O(x^{1/2} \log^{-1} x),$$

since for $t \leq 0$ we have $w \leq 1$, and partial summation gives

$$\sum_{n > x} h(n, t) n^{-1} = O(x^{-1/2} \log^{-1} x).$$

Now take $y = x/\log x$, $z = \log x$. From $\sum_{n=1}^{\infty} f^t(n) n^{-s} = \zeta(s)g(s, t)$ we get

$$\begin{aligned} \sum_{n \leq x} f^t(n) &= \sum_{mn \leq x} h(n, t) = \sum_{n \leq y} h(n, t)[x/n] + \sum_{m \leq z} 1 \sum_{n \leq x/m} h(n, t) \\ &\quad - \sum_{m \leq z} 1 \sum_{n \leq y} h(n, t) = S_1 + S_2 - S_3. \end{aligned}$$

$$S_3 = O(zy^{1/2} \log^{-1} x) = O(x^{1/2} \log^{-1/2} x).$$

$$S_2 = O(x^{1/2} \sum_{m \leq z} m^{-1/2} \log^{-1} x/m) = O(x^{1/2} \log^{-1} y \sum_{m \leq z} m^{-1/2}) = O(x^{1/2} \log^{-1/2} x).$$

$$\begin{aligned} S_1 &= \sum_{n \leq y} h(n, t)(x/n + O(1)) = C(t)x + x \sum_{n > y} h(n, t) n^{-1} + O\left(\sum_{n \leq y} |h(n, t)|\right) \\ &= C(t)x + O(xy^{-1/2} \log^{-1} y) + O(y^{1/2}) = C(t)x + O(x^{1/2} \log^{-1/2} x), \end{aligned}$$

so that we obtain uniformly in t

$$(7) \quad \sum_{n \leq x} f^t(n) = C(t)x + O(x^{1/2} \log^{-1/2} x),$$

where

$$C(t) = g(1, t) = \sum_{n=1}^{\infty} h(n, t) n^{-1} = \prod_p \left(1 + \sum_{k=2}^{\infty} (g^t(k) - g^t(k-1)) p^{-k}\right).$$

Putting (7) into (3) and integrating from $-T$ to $0(T > 0)$ we get

$$(8) \quad \sum'_{n \leq x} 1/\log f(n) = x \int_{-T}^0 (C(t) - 6/\pi^2) dt + 0(x^{1/2} \log^{-1/2} x \cdot T) \\ + 0(Tx^{1/2} \exp(-C \log^{3/5} x (\log \log x)^{-1/5})) + \sum'_{n \leq x} f^{-T}(n)/\log f(n).$$

To estimate $C(t) - 6/\pi^2$ for $t \leq 0$, let $C(t) = \prod_p (1 - p^{-2} + u(p, t))$, where

$$0 < u(p, t) = g^t(2)p^{-2} + (g^t(3) - g^t(2))p^{-3} + (g^t(4) - g^t(3))p^{-4} + \dots \\ = g^t(2)(p^{-2} - p^{-3}) + g^t(3)(p^{-3} - p^{-4}) + g^t(4)(p^{-4} - p^{-5}) + \dots \\ \leq g^t(r)(p^{-2} - p^{-3} + p^{-3} - p^{-4} + p^{-4} - \dots) = g^t(r)p^{-2},$$

where r is an integer such that $g(\nu) \geq g(r) > 1$ for $\nu = 2, 3, \dots$. Such an integer certainly exists, since $\liminf_{\nu \rightarrow \infty} g(\nu) > 1$.

Using the inequality $\log(x + y) \leq \log x + y/x$ ($x, y > 0$) we get

$$C(t) = \exp\left(\log \prod_p (1 - p^{-2} + u(p, t))\right) = \exp\left(\sum_p \log(1 - p^{-2} + u(p, t))\right) \\ \leq \exp\left(\sum_p \log(1 - p^{-2}) + \sum_p (1 - p^{-2})^{-1} u(p, t)\right) \\ \leq \frac{6}{\pi^2} \exp\left(g^t(r) \sum_p (p^2 - 1)^{-1}\right) \leq 6 \exp(g^t(r))/\pi^2.$$

If $t < 0$ is small enough we get

$$(9) \quad 0 \leq C(t) - 6/\pi^2 \leq (6/\pi^2)(\exp(g^t(r)) - 1) = 0(g^t(r)),$$

$$(10) \quad \int_{-\infty}^{-T} (C(t) - 6/\pi^2) dt = 0\left(\int_{-\infty}^{-T} g^t(r) dt\right) = 0(g^{-T}(r)).$$

If $n = p_1^{\nu_1} \cdots p_i^{\nu_i}$, then $f(n) = g(\nu_1) \cdots g(\nu_i) \geq g(r) > 1$ if $f(n) > 1$, so that $f^T(n) \log f(n) \geq g^T(r) \log g(r)$ if $f(n) > 1$, and we obtain

$$(11) \quad \sum'_{n \leq x} f^{-T}(n)/\log f(n) \leq \sum'_{n \leq x} g^{-T}(r)/\log g(r) = 0\left(g^{-T}(r) \sum_{n \leq x} 1\right) = 0(g^{-T}(r)x).$$

Writing $\int_{-T}^0 (C(t) - 6/\pi^2) dt = \int_{-\infty}^0 - \int_{-\infty}^{-T}$ and using (10) and (11) we get from (8)

$$(12) \quad \sum'_{n \leq x} 1/\log f(n) = x \int_{-\infty}^0 (C(t) - 6/\pi^2) dt + 0(g^{-T}(r)x) + 0(x^{1/2} \log^{-1/2} x \cdot T) \\ + 0(Tx^{1/2} \exp(-C \log^{3/5} x (\log \log x)^{-1/5})).$$

Now take $T = \log x/2 \log g(r)$. Then we have

$$g^{-T}(r)x = \exp(-T \log g(r) + \log x) = \exp(\frac{1}{2} \log x) = x^{1/2},$$

and so the theorem is proved.

As a first example, let us take $a(n)$, the number of non-isomorphic abelian groups of order n . It is well-known (see [4]) that $a(n)$ is multiplicative, and that $a(p^\nu) = P(\nu)$ for any prime p and $\nu = 1, 2, \dots$, where $P(\nu)$ is the number of unrestricted partitions of the integers ν , so that $P(\nu) = 1$ if $\nu = 1$ and $P(\nu)$ is strictly increasing with ν . Therefore the conditions of our theorem are satisfied, and (1) holds with $f(n) = a(n)$, $g(k) = P(k)$. Note that in this case we have $\liminf_{\nu \rightarrow \infty} g(\nu) = +\infty$ and $r = 2$, $g(r) = 2$.

Examples of other multiplicative, prime-independent functions that satisfy the conditions of our theorem may be readily found among enumerative functions of certain algebraic structures. Such is for example (see [6] for a detailed discussion) $S(n)$, the number of non-isomorphic semisimple finite rings of order n .

Finally let us consider $\tau^{(e)}(n)$, the number of exponential divisors of n . A divisor $d = p_1^{b_1} \cdots p_i^{b_i}$ is called an exponential divisor of $n = p_1^{\nu_1} \cdots p_i^{\nu_i}$ if $b_1 \mid \nu_1, \dots, b_i \mid \nu_i$ (see [8]). It follows that $\tau^{(e)}(n)$ is a multiplicative, prime-independent arithmetical function for which $\tau^{(e)}(p^\nu) = \tau(\nu)$, where $\tau(\nu)$ is the ordinary number of divisors function. Since $\tau(1) = 1$ and $\tau(\nu) \geq 2$ if $\nu \geq 2$, the conditions of our theorem are satisfied and (1) holds with $f(n) = \tau^{(e)}(n)$ and $g(k) = \tau(k)$. Again it is of interest to note that $\liminf_{\nu \rightarrow \infty} g(\nu) = 2$ and $r = 2$, $g(r) = 2$ also.

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