SOME REMARKS ON ADDITIVE FUNCTIONS

By

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§ 1. Introduction.

An arithmetic function f is said to be additive if $f(mn) = f(m) + f(n)$, when $(m, n) = 1$. Sums of the form $\sum'_{n \leq x} 1/f(n)$ have already been studied [1], [3], (here Σ' denotes summation over those values of n for which $f(n) \neq 0$). In a recent paper [2], we have obtained asymptotic expansions for $\sum'_{n \leqslant x} \Omega(n)/\omega(n)$, where the additive functions $\Omega(n)$ and $\omega(n)$ denote the total number of prime factors of n and the number of distinct prime factors of n, respectively. Our estimate was based on a deep analytic result of A. Selberg [6].

The purpose of this paper is to study sums of quotients of additive functions for different classes of functions. Also we shall point out a more elementary method which permits us, in certain cases, to obtain the first two terms of the asymptotic expansion of $\sum'_{n \leqslant x} g(n)/f(n)$; the method is essentially due to Janos Galambos*.

§ 2. Notation.

$\zeta(w)$ denotes the Riemann Zeta Function. The numbers $c, c_1, c_2, \ldots$ denote absolute positive constants. The numbers r and s denote positive integers. Finally p and q stand for prime numbers.

§ 3. Additive functions which "behave like $c \log \log n$".

Theorem 1. Let g and f be two additive functions for which there exist two positive constants a and b such that the two series $\sum_p \frac{f(p) - a}{p}$ and $\sum_p \frac{g(p) - b}{p}$ converge absolutely; also assume that

*Private communication.

$$\sum_{\substack{n \leqslant x \\ f(n) = 0}} 1 \quad = 0\,(x/(\log x)\,), \qquad \ldots(1)$$

$$\sum_{\substack{n \leqslant x \\ f(n) = 0}} g(n) \quad = 0\,(x/(\log\log x)^{\frac{1}{2}}), \qquad \ldots(2)$$

$$\sum_{\substack{n \leqslant x \\ f(n) = 0}} (g(n)\,)^2 = 0\,(x) \qquad \ldots(3)$$

and that $| f(p^r) | < c_1 r$ and $| g(p^r) | < c_1 r$, for all primes p and integers $r \geq 1$, where c_1 depends only on f and g, Finally suppose that the function f satisfies the additional condition.

$$f(n) \neq 0 \Rightarrow f(n) \geq c_2, \text{ for each integer } n \geq 1.$$

Then

$$\sum_{n \leqslant x}{}' \frac{g(n)}{f(r)} = \frac{bx}{a}\left\{ 1 + \frac{d}{\log\log x} + 0(1/(\log\log x)^{\frac{3}{2}} \right\}, \ldots(4)$$

with $d = \sum_{r=1}^{\infty}{}' \sum_{p} (1 - p^{-1})\,(\,(g\,(p^r) - f\,(p^r)\,)/p^r)$.

Proof. First of all, not• that, without loss in generality, we can assume that $a = b = 1$, for otherwise consider $f_0(n) = a^{-1} f(n)$ and $g_0(n) = b^{-1}\, g(n)$, and then we have $\sum_{n \leqslant x}{}' \frac{g(n)}{f(n)} = \frac{b}{a} \sum_{n \leqslant x}{}' \frac{g_0(n)}{f_0(n)}$.

Now let $D(n) = g(n) - f\,(n)$ and, for $f(n) \neq 0$, $h(n) = (g\,(n)/f(n) - 1)\log\log n$. We will show that

$$\frac{1}{x} \sum_{n \leqslant x}{}' h(n) = \frac{1}{x} \sum_{n \leqslant x}{}' D(n) + 0(1/(\log\log x)^{\frac{1}{2}}),$$

from which (4) will follow by partial summation and by the use of relations (1) and (2).

Now clearly, when $f(n) \neq 0$,

$$h(n) = D(n) \quad \frac{\log\log n}{f(n)} = D(n) + D(n)\left\{\frac{\log\log n}{f(n)} - 1\right\}.$$

Thus

$$\sum_{n \leqslant x}' h(n) = \sum_{n \leqslant x}' D(n) + \sum_{n \leqslant x}' D(n) \left\{ \frac{\log \log n}{f(n)} - 1 \right\}$$

We want to prove that

$$\frac{1}{x} \sum_{n \leqslant x}' D(n) \left\{ \frac{\log \log n}{f(n)} - 1 \right\} = 0(1/(\log \log x)^{\frac{1}{2}}). \quad \ldots(5)$$

But, by Schwartz inequality, we have

$$\left[\sum_{n \leqslant x}' D(n) \left\{ \frac{\log \log n}{f(n)} - 1 \right\} \right]^2$$

$$\leqslant \sum_{n \leqslant x}' D^2(n) \sum_{n \leqslant x}' \left(\frac{\log \log n - f(n)}{f(n)} \right)^2$$

$$\leqslant \sum_{n \leqslant x}' D^2(n) \left\{ \sum_{n \leqslant x}' (\log \log n - f(n))^4 \sum_{n \leqslant x}' 1/(f(n))^4 \right\}^{\frac{1}{2}} \quad (6)$$

Hence

$$\sum_{n \leqslant x}' D(n) \left\{ \frac{\log \log n}{f(n)} - 1 \right\}$$

$$\leqslant \sqrt{\sum_{n \leqslant x}' D^2(n) \sqrt{\sum_{n \leqslant x}' (\log \log n - f(n))^4 \sum_{n \leqslant x}' 1/(f(n))^4}}$$

Assume for the moment that the following relations are true:

$$\frac{1}{x} \sum_{n \leqslant x}' D^2(n) = 0(1) \quad (7)$$

$$\sum_{n \leqslant x}' (\log \log n - f(n))^4 = 0(x(\log \log x)^2) \quad (8)$$

$$\sum_{n \leqslant x}' 1/(f(n))^4 = 0(x/(\log \log x)^4) \quad (9)$$

Then, substituting these in (6), we obtain (5) and our theorem is proved. It remains to prove (7), (8) and (9).

First observe that

$$\sum_{n \leqslant x} D^2(n) = \sum_{n \leqslant x} \left(\sum_{p^r \| n} D(p^r) \right)^2 = \sum_{n \leqslant x} \left(\sum_{p^r | n} \left(D(p^r) - D(p^{r-1}) \right) \right)^2$$

$$= \sum_{n \leqslant x} \sum_{\substack{p^r \mid n \\ q^s \mid n}} (D(p^r) - D(p^{r-1}))(D(p^s) - D(p^{s-1}))$$

$$= \sum_{\substack{p^r \leqslant x \\ q^s \leqslant x \\ p \neq q}} (D(p^r) - D(p^{r-1}))(D(q^s) - D(q^{s-1})) \left[\frac{x}{p^r q^s}\right]$$

$$+ \sum_{p^s \leqslant p^r \leqslant x} (D(p^r) - D(p^{r-1}))(D(p^s) - D(p^{s-1})) \left[\frac{x}{p^r}\right]$$

$$\leqslant x \left\{ \sum_{\substack{p^r \leqslant x \\ q^s \leqslant x}} \frac{|D(p^r) - D(p^{r-1})|}{p^r} \frac{|D(q^s) - D(q^{s-1})|}{q^s} + \sum_{p^s \leqslant p^r \leqslant x} \frac{|D(p^r) - D(p^{r-1})|}{p^r} |D(p^s) - D(p^{s-1})| \right\}$$

Now using the fact that $\sum_{p} \frac{D(p)}{p}$ converges absolutely and the fact that $|D(p^r) - D(p^{r-1})| \leqslant 4c_1 r$, for each integer $r \geq 1$, one easily obtains that $\sum_{n \leqslant x} D^2(n) = 0(x)$, which together with relation (3) proves relation (7).

In order to prove (8), it is sufficient to prove that

$\sum_{n \leqslant x} (f(n) - \log\log x)^4 = 0(x(\log\log x)^2)$. Now write

$$\sum_{n \leqslant x} (f(n) - \log\log x)^4 = \sum_{i=0}^{4} (-1)^i \binom{4}{i} (\log\log x)^i \left(\sum_{n \leqslant x} (f(n))^{4-i} \right). \qquad (10)$$

Then we proceed to evaluate separately $\sum_{n \leqslant x} (f(n))^j$ for $j = 1, 2, 3, 4$, and by using the fact that

$$\sum_{p \leqslant x} \frac{f(p)}{p} = \sum_{p \leqslant x} \frac{f(p) - 1}{p} + \sum_{p \leqslant x} \frac{1}{p}$$

$$= \sum_{p \leqslant x} \frac{f(p) - 1}{p} + \log\log x + c_3 + 0(x/(\log x))$$

and that $|f(p^r)| < c_1 r$, for each integer $r \geq 1$, it follows, after some computations, that the right member of (10) becomes $0(x(\log\log x)^2)$, as desired.

Finally relation (9) is easily established by observing that $f(n) = \sum_{p^r || n} f(p^r) \geqslant c_2 \sum_{p^r || n} 1 = c_2\,\Omega(n)$, when $f(n) \neq 0$, and then proceeding as in § 2 of [3].

§ 4. Additive functions which "behave like $c \log n$".

We now consider a different class of additive functions. In fact we prove the folowing.

Theorem 2. Let g and f be two additive functions for which there exist two non-zero constants a and b such that, for each prime p and integer $r \geq 1$,

$$g(p^r) = b\, r \log p + R_g(p^r)$$

$$\text{and} \quad f(p^r) = a\, r \log p + R_f(p^r),$$

with $|R_h(p^r) - R_h(p^{r-1})| < c_4/p^{r\lambda}$, for some $\lambda > 0$, uniformly in $r \geq 1$, when $h = g$ or $h = f$. Also assume for simplicity that $f(n) \neq 0$, for all integers $n \geqq 2$.

Then, given any positive integer α, we have

$$\sum_{n \leqslant x}{}' \frac{g(n)}{f(n)} = \frac{bx}{a}\left\{1 + \sum_{i=1}^{\alpha} \frac{e_i}{(\log x)^i} + 0\left(\frac{1}{(\log x)^{\alpha+1}}\right)\right\}, \quad (11)$$

where the e_i's are computable constants.

Proof. As in the proof of Theorem 1, we can suppose that $a = b = 1$.

We first proceed to estimate $\sum_{n \leqslant x} g(n)^u f(n)^t$, for $u \in \mathbb{C}$, $|u| \leqslant \eta < 1/4$ and $t \in [-\frac{1}{4}, 0]$. Define $g_1(n) = e^{g(n)}$ and $f_1(n) = e^{f(n)}$. Then $g_1(n)$ and $f_1(n)$ are multiplicative, which implies that, for $w \in \mathbb{C}$ and $\mathrm{Re}(w) > 1$, we have

$$\sum_{n=1}^{\infty} \frac{(g_1(n)n^{-1})^u (f_1(n)n^{-1})^t}{n^w} = \prod_p \left(1 + \frac{(g_1(p)p^{-1})^u (f_1(p)p^{-1})^t}{p^w} + \dots \right)$$

$$= \zeta(w) \prod_p (1 - p^{-w}) \prod_p (1 + \dots)$$

$$= \zeta(w) \sum_{n=1}^{\infty} \frac{h(n, u, t)}{n^w}$$

$$= \zeta(w)\, G(w\,;\, u, t), \text{ say.}$$

Using the conditions on f and g, one can easily prove that $G(1\,;\, u, t)$ is absolutely convergent uniformly in u and t. This permits us to apply formula (1.10.22) on p. 47 of [5] and obtain

$$\sum_{n \leqslant x} (g_1(n)n^{-1})^u (f_1(n)n^{-1})^t = x \sum_{n \leqslant x} \frac{h(n, u, t)}{n} + 0(x^{\frac{1}{2}} \max_{d \leqslant x^{\frac{1}{2}}} | H(d, u, t) |). \tag{12}$$

where the constant in $0(\dots)$ is independant of u and t, and

$$H(d, u, t) = \sum_{n \leqslant d} h(n, u, t).$$

We now show that (12) can be replaced by

$$\sum_{n \leqslant x} (g_1(n)n^{-1})^u (f_1(n)n^{-1})^t = x\, G(1\,;\, u, t) + 0(x^{1-\varepsilon}),$$

uniformly in u and t, for some $\varepsilon > 0$.

For this, we observe that

$$| h(p, u, t) | = | (g_1(p)p^{-1})^u (f_1(p)p^{-1})^t - 1 |$$

$$= | \exp(R_g(p)u + R_f(p)t) - 1 |$$

$$\leqslant | R_g(p)u + R_f(p)t | + \frac{| \dots |^2}{2\,!} + \dots$$

$$\leqslant | R_g(p) | + | R_f(p) | + \frac{(| R_g(p) | + | R_f(p) |)^2}{2\,!} + \dots$$

$$< 2\, c_4\, p^{-\lambda} + \frac{(2\, c_4\, p^{-\lambda})^2}{2\,!} + \dots$$

$$< 6\, c_4\, p^{-\lambda}$$

Now, for $r \geq 2$

$$
\begin{aligned}
| h(p^r, u, t) | &= | (g_1(p^r)p^{-r})^u f_1(p^r)p^{-r})^t - (g_1(p^{r-1})p^{-r+1})^u (f_1(p^{r-1})p^{-r+1})^t | \\
&= | \exp (R_g(p^r)u + R_f(p^r)t) - \exp (R_g(p^{r-1})u + R_f(p^{r-1})t) | \\
&= | (\exp (\ldots) - 1) - (\exp (\ldots) - 1) | \\
&= | R_g(p^r)u + R_f(p^r)t) + \frac{(\ldots)^2}{2\,!} + \ldots - (R_g (p^{r-1})u \\
&\qquad + R_f(p^{r-1})t) - \frac{(\ldots)^2}{2\,!} - \ldots | \\
&= | (R_g(p^r) - R_g(p^{r-1}))u + (R_f(p^r) - R_f(p^{r-1}))t \\
&\quad + | \frac{(R_g(p^r)u + R_f(p^r)t)^2 - (R_g(p^{r-1})u + R_f(p^{r-1})t)^2}{2\,!} + \ldots 1 \\
&\leq | (R_g(p^r) - R_g(p^{r-1}) | + | R_f(p^r) - R_f(p^{r-1}) | \\
&\quad + \left| \frac{(R_g(p^r) + R_f(p^r))^2 + (R_g(p^{r-1}) + R_f(p^{r-1}))^2}{2\,!} \right| + \ldots \\
&< \frac{2\,c_4}{p^{r\lambda}} + \frac{2}{2!}\left(\frac{2\,c_4}{p^{r\lambda}}\right)^2 + \frac{2}{3!}\left(\frac{2\,c_4}{p^{r\lambda}}\right)^3 + \ldots \\
&< \frac{12\,c_4}{p^{r\lambda}}
\end{aligned}
$$

Hence, since $h(n, u, t)$ is multiplicative, we have

$$
\begin{aligned}
| H(d, u, t) | &= | \sum_{n \leqq d} h(n, u, t) | \leq \sum_{n \leq d} | h(n, u, t) | \\
&= \sum_{n \leq d} \prod_{p^r || n} | h(p^r, u, t) | < \sum_{n \leq d} \prod_{p^r || n} \frac{12\,c_4}{p^{r\lambda}} \\
&= \sum_{n \leq d} (12\,c_4)^{\Omega(n)} \prod_{p^r || n} \frac{1}{p^{r\lambda}} \\
&= \sum_{n \leq d} (12\,c_4)^{\Omega(n)} n^{-\lambda}.
\end{aligned}
$$

But $\sum_{n \leq d} (12\,c_4)^{\Omega(n)}$ can be shown to be $0(d\,(\log d)^{12\,c_4 - 1})$ (see [4]).

Hence, by partial summation, we have that

$$
\sum_{n \leqslant d} (12\,c_4)^{\Omega(n)} n^{-\lambda} = 0(d^{1-\lambda} (\log d)^{12\,c_4 - 1}) = 0(d^{1 - \frac{\lambda}{2}}).
$$

Therefore

$$\operatorname*{Max}_{d \leqslant \sqrt{x}} | H(d, u, t) | = 0(x^{\frac{1}{2} - \frac{\lambda}{4}}).$$

Finally

$$\left| \sum_{n > x} \frac{h(n, u, t)}{n} \right| \leq \sum_{n > x} \left| \frac{h(n, u, t)}{n} \right|$$

$$= \sum_{n > x} \left| \frac{H(n, u, t) - H(n-1, u, t)}{n} \right| \leq \sum_{n > x} \frac{| H(n, u, t) |}{n(n+1)}$$

$$\leq \sum_{n > x} \frac{| H(n, u, t) |}{n^2} \leq \sum_{n > x} \frac{c_5 n^{1 - \frac{\lambda}{2}}}{n^2}$$

$$= c_5 \sum_{n > x} n^{-1 - \frac{\lambda}{2}} = 0(x^{-\frac{\lambda}{2}}). \qquad (14)$$

Hence, using (13) and (14), we can replace (12) by

$$\sum_{n \leq x} (g_1(n)\, n^{-1})^u (f_1(n)\, n^{-1})^t = \sum_{n=1}^{\infty} \frac{h(n, u, t)}{n} + 0(x^{1-\varepsilon}),$$

for some $\varepsilon > 0$, uniformly for $t \in [-\frac{1}{4}, 0]$ and $| u | \leq \eta$.
This implies that

$$\sum_{n \leq x} g_1(n)^u f_1(n)^t n^{-u-t} = x\, G(1 ; u, t) + 0(x^{1-\varepsilon}),$$

and thus

$$\sum_{n \leqslant x} g_1(n)^u f_1(n)^t = \sum_{n \leqslant x} g_1(n)^u f_1(n)^t n^{-u-t} n^{u+t}$$

$$= G(1 ; u, t)\, x^{1+u+t} + 0(x^{1-\varepsilon+u+t}) - \int_1^x (G(1;u,t)y)(u+t)y^{u+t-1}\, dy$$

$$+ 0\left(\int_1^x y^{1-\varepsilon} (u+t)\, y^{u+t-1}\, dy\right)$$

$$= \left(G(1 ; u, t) - \frac{G(1;u,t)\,(u+t)}{u+t+1} \right) x^{u+t+1} + 0(x^{1-\varepsilon+u+t}) + \frac{(u+t)}{u+t+1}$$

$$= \frac{G(1 ; u, t)}{u+t+1}\, x^{u+t+1} + 0(x^{1-\varepsilon+u+t})$$

Now

$$\sum_{n \leqslant x}{}' g_1(n)^u f_1(n)^t g(n) = \left[\frac{d}{du} \frac{G(1\,;u,t)}{u+t+1}\right] x^{u+t+1}$$

$$+ \frac{G(1\,;u,t)}{u+t+1} x^{u+t+1} \log x + 0(x^{1-\varepsilon+u+t} \log x)$$

where we made use of Cauchy's inequality to estimate the error term. Setting $u=0$ in this last relation, we obtain

$$\sum_{n \leqslant x} g(n) f_1(n)^t = \frac{d}{du} \frac{G(1\,;u,t)}{u+t+1}\Big|_{u=0} x^{t+1} + \frac{G(1;0,t)}{t+1} x^{t+1} \log x$$

$$+ 0(x^{1-\varepsilon+t} \log x)$$

$$= Z(t)\, x^{t+1} + \frac{G(1;0,t)}{t+1} x^{t+1} \log x + 0(x^{1-\varepsilon+t} \log x) \quad (15)$$

where we have written $Z(t)$ for $\frac{d}{du} \frac{G(1\,;u,t)}{u+t+1}\Big|_{u=0}$

Then observe that

$$\int_{-\frac{3}{4}}^{0} \Big(\sum_{n \leqslant x}{}' g(n) f_1(n)^t\Big)\, dt = \sum_{n \leqslant x}{}' \frac{g(n)}{f(n)} f_1(n)^t \Big|_{-3/4}^{0}$$

$$= \sum_{n \leqslant x}{}' \frac{g(n)}{f(n)} - \sum_{n \leq x}{}' \frac{g(n)}{f(n)} f_1(n)^{-3/4}, \quad (16)$$

but by the conditions on f and g, we gave

$$\Big|\sum_{n \leqslant x}{}' \frac{g(n)}{f(n)} f_1(n)^{-3/4}\Big| \leq c_6 \sum_{n \leqslant x}{}' f_1(n)^{-3/4}$$

$$= c_6 \sum_{n \leq x}{}' \frac{\exp\Big((-3/4) \sum_{p^r \| n} R_f(p^r)\Big)}{n^{3/4}}$$

$$\leqslant c_6 \sum_{n \leqslant x}{}' n^{-3/4} \exp\Big((3/4)\, c_7 \sum_{p^r \| n} r\, p^{-r\lambda}\Big) = 0(x^{1/4}). \quad (17)$$

The fact that this last expression is $0(x^{1/4})$ can be seen as follows. Let

$$k(n) = \exp\Big((3/4)\, c_7 \sum_{p^r \| n} r\, p^{-r\lambda}\Big),$$

then $k(n)$ is easily seen to be multiplicative and observing that $\sum_{n=1}^{\infty} k(n)\, n^{-w} = \zeta(w)\, l(w)$, where $l(1) = 0(1)$, we can use Renyi's theorem (see [5]) and obtain that $\sum_{n \leqslant x} k(n) = 0(x)$, hence

$$\sum_{n \leqslant x}' k(n)\, n^{-3/4} = 0(x^{1/4}).$$

Now taking into account (16) and (17), we integrate both sides of (15) between $-3/4$ and 0, and using essentially the same procedure as in [3], we have

$$\sum_{n \leqslant x}' \frac{g(n)}{f(n)} = \int_{-3/4}^{0} Z(t)\, x^{t+1}\, dt + \int_{-3/4}^{0} \frac{G(1;0,t)}{t+1}\, x^{t+1} (\log x)\, dt + 0(x^{1-\varepsilon})$$

$$= x \left\{ \frac{Z(t)\, x^t}{\log x} \Bigg|_{-3/4}^{0} - \frac{Z'(t)\, x^t}{(\log x)^2} \Bigg|_{-3/4}^{0} + \dots \right.$$

$$\left. + \frac{G(1;0,t)\, x^t (\log x)}{(t+1)(\log x)} \Bigg|_{-3/4}^{0} - \left(\frac{G(1;0,t)}{t+1}\right)' \frac{x^t (\log x)}{(\log x)^2} \Bigg|_{-3/4}^{0} + \dots + 0(x^{-\varepsilon}) \right\}$$

$$= x \left\{ G(1;0,0) + \frac{1}{\log x} \left(Z(0) - \left(\frac{G(1;0,t)}{t+1}\right)' \Bigg|_{t=0} \right) - \dots + 0(x^{-\varepsilon}) \right\},$$

which establishes relation (11), and therefore our theorem is proved.

Now applying the method of proof of Theorem 1 to the class of functions stated in Theorem 2, this time taking

$$h(n) = (g(n)/f(n) - 1) \log n :$$

we obtain the following estimate :

$$\sum_{n \leqslant x}' \frac{g(n)}{f(n)} = \frac{bx}{a} \left\{ 1 + \frac{d}{\log x} + 0\left(\frac{\log \log x}{(\log x)^2} \right) \right\}$$

where the relative improvement in the error term comes from the following three estimates, analogue to relations (7), (8) and (9) of Theorem 1 :

$$\frac{1}{x} \sum_{n \leqslant x}' D^2(n) = 0(1),$$

$$\sum_{n \leqslant x}' (\log n - f(n))^4 = 0(x(\log \log x)^4)$$

$$\sum_{n \leqslant x}' 1/(f(n))^4 = 0(x/(\log x)^4).$$

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