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Sums of reciprocals of additive functions

by

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1. Summary. In the present paper we shall investigate the problem of finding asymptotic expansions for sums of reciprocals of additive functions, in the light of probabilistic number theory. Our first observation is that for a large class of functions $f(n)$, termed as prime independent, the asymptotic formula

(1) $$\sum_{n\leqslant x}' 1/f(n) = x[1+o(1)][f(2)\log\log x]^{-1}$$

is immediate from known results, where $\sum'$ denotes summation over those values of n for which $f(n) \neq 0$. We shall then point out the significance of obtaining a second term on the right hand side of (1). The first part of our paper is therefore a complement to the recent work [1] of the first named author. In the second part, an asymptotic expansion is given for the sum of the reciprocals of $\log\sigma(n)$, where $\sigma(n)$ is the sum of the divisors of n. This part has two essentially distinct features from the paper [1]. First of all, no reference is made to any deep result, and secondly, the expansion is in powers of $1/\log x$. The discussion in the first part will reveal that the increase from $\log\log x$ to $\log x$ in the asymptotic expansion results, in general, in additional difficulty.

2. Sums of reciprocals and probabilistic number theory. Let $\omega(n)$ denote the number of different prime factors of n. A classical theorem of Hardy and Ramanujan states that "for almost all" n, $\omega(n) \sim \log\log n$. By giving an accurate meaning to the expression "almost all", we shall be able to obtain the desired asymptotic formula (1) for the case $f(n) = \omega(n)$. Since the theorem of Hardy and Ramanujan has several extensions, we are able to start with a general set up. Our class of functions $f(n)$ can include, roughly speaking, all functions for which $f(n) \sim \log\log n$ for "almost all" n. This is made possible by the function $\log\log n$ being slowly varying in the sense that

(2) $$\log\log n = \log\log x + O(1), \qquad x^{1/2} \leqslant n \leqslant x,\ x\to+\infty.$$

Let now $f(n)$ be a given arithmetical function and let $R(x)$ be a positive function tending to $+\infty$. Let $A(x)$ denote the number of positive integers $n \leqslant x$ for which the inequalities

$$\log\log n - R(x) \leqslant f(n) \leqslant \log\log n + R(x) \tag{3}$$

fail to hold. In view of (2), for $f(n) \geqslant 1$ for all those values of n when $f(n) \neq 0$, we evidently have

$$\sum_{n\leqslant x}{}' \frac{1}{f(n)} = \sum_{n\leqslant\sqrt{x}}{}' + \sum_{\sqrt{x}<n\leqslant x}{}' \leqslant x^{1/2} + A(x) + \frac{x - x^{1/2} - A(x)}{\log\log x} \times \left[1 + O\left(\frac{R(x)}{\log\log x}\right)\right], \tag{4}$$

and

$$\sum_{n\leqslant x}{}' \frac{1}{f(n)} \geqslant \sum_{\sqrt{x}\leqslant n\leqslant x}{}' \geqslant \frac{x - x^{1/2} - A(x) + A(x^{1/2})}{\log\log x} \left[1 + O\left(\frac{R(x)}{\log\log x}\right)\right]. \tag{5}$$

(4) and (5) imply (1) for any function $f(n)$ having the property that

$$A(x) = o(x/\log\log x) \quad \text{with} \quad R(x) = o(\log\log x).$$

This is known to hold for a large class of functions. As a matter of fact, the large deviation theorem of Kubilius [5] on p. 161, together with the remark on p. 168, implies that, when (3) is applied to $f(n)/f(2)$, where $f(n)$ is additive with $f(p) = f(2) \neq 0$ for all primes p,

$$A(x) = o\left(\frac{x}{(\log\log x)^2}\right) \quad \text{if} \quad R(x) = (\log\log x)^{1/2+\varepsilon}, \tag{6}$$

where ε is an arbitrary fixed positive constant. Applying (4), (5) and (6) to the function $f(n)/f(2)$, $f(2) \neq 0$, satisfying the assumption above, we get

$$\sum_{n\leqslant x}{}' \frac{1}{f(n)} = \frac{x}{f(2)\log\log x} + O\left(\frac{x}{(\log\log x)^{3/2-\varepsilon}}\right). \tag{7}$$

(7) is a somewhat stronger statement than (1) by specifying the error term. We add that the error term can be slightly improved on this line of attack, but the order of magnitude $x/(\log\log x)^{3/2}$ can not be achieved as it easily follows from the asymptotic normality of

$$\big(f(n) - f(2)\log\log n\big)/f(2)(\log\log n)^{1/2}$$

(see [5], p. 61 and the inequality in [2]). Therefore, essential new information can only be achieved by determining the exact order of magnitude of the second term on the right hand side of (7). The finite asymptotic expansion obtained in [1] goes further than this.

We remark that asymptotic formulas, similar to (1), can easily be obtained by the present approach for sums of ratios $f(n)/g(n)$, if both $f(n)$ and $g(n)$ are 'close' to $\log\log n$ (in the sense of the previous discussion). (1) can also be extended to the case when the argument n goes through given sequences of integers (others than the consecutive ones). This is made possible by the probabilistic approach of the second named author to distribution problems of arithmetical functions; see [3], [4]. For this case, however, a much weaker error term is obtained than the one in (7).

A major tool in proving (7) was the property (2) of the function $\log\log n$. When the role of $\log\log n$ is replaced by a function growing too rapidly to $+\infty$, our method may fail. To determine the order of magnitude of the sum in (1) remains, however, a simple problem for most functions.

3. On the sum of the divisors of n. Let $\sigma(n)$ denote the sum of the divisors of n. It is a classical result of probabilistic number theory that $\log\sigma(n) = \log n + O(1)$, for "almost all n", see, e.g., [5], p. 74. D. Rearick asked (personal communication) if an asymptotic expansion is valid for $\sum_{2\leqslant n\leqslant x} 1/\log\sigma(n)$. The asymptotic formula

$$\sum_{2\leqslant n\leqslant x} 1/\log\sigma(n) = [1+o(1)]x/\log x$$

can be established by the approach of the previous section. The error term by this approach, however, can not be properly estimated. On the other hand, by an analytical method, we get the following asymptotic expansion.

THEOREM. *Let a be an arbitrary positive integer. Then*

$$\sum_{n=2}^{x} \frac{1}{\log\sigma(n)} = x\sum_{j=1}^{a} \frac{a_j}{(\log x)^j} + O\left(\frac{x}{(\log x)^{a+1}}\right),$$

where $a_1 = 1$ and, in general,

$$a_j = (-1)^{j-1} E^{(j-1)}(t)|_{t=0},$$

with

$$E(t) = \frac{1}{t+1}\prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{j=1}^{+\infty}\frac{1}{p^j}\left(1+\frac{1}{p}+\ldots+\frac{1}{p^j}\right)^t\right), \qquad p \text{ prime}.$$

Proof. Let us first estimate $\sum_{2\leqslant n\leqslant x} \sigma(n)^t$ for $t<0$. By the elementary property of $\sigma(n)$ being multiplicative, we have the following product representation of the Dirichlet series of $[\sigma(n)/n]^t$ for $s = u+iv$, $u>1$

$$\sum_{n=1}^{+\infty} \frac{\sigma^t(n)n^{-t}}{n^s} = \prod_p \left\{1+\sum_{j=1}^{+\infty}\frac{1}{p^{js}}\left(1+\frac{1}{p}+\ldots+\frac{1}{p^j}\right)^t\right\}.$$

Dividing by the Riemann zeta function, we get

$$\sum_{n=1}^{+\infty} \frac{\sigma^t(n)n^{-t}}{n^s} = \zeta(s)\prod_p \left(1-\frac{1}{p^s}\right)\left\{1+\sum_{j=1}^{+\infty}\frac{1}{p^{js}}\left(1+\frac{1}{p}+\ldots+\frac{1}{p^j}\right)^t\right\}$$

$$= \zeta(s)\sum_{n=1}^{+\infty}\frac{h(n,t)}{n^s} = \zeta(s)g(s,t), \text{ say.}$$

Since $t<0$, $g(1,t)$ is absolutely convergent. By the formula (1.10.29) on p. 47 of [6], we therefore have

$$\sum_{2\leqslant n\leqslant x} \sigma^t(n)n^{-t} = x\sum_{n=1}^{x}\frac{h(n,t)}{n} + O\left(x^{1/2}\max_{d\leqslant x}|H(d,t)|\right), \tag{8}$$

where the constant in $O(\ldots)$ is independent of t, and

$$H(d,t) = \sum_{n\leqslant d} h(n,t).$$

We now show that (8) can be extended to

$$\sum_{2\leqslant n\leqslant x} \sigma^t(n)n^{-t} = xg(1,t)+O(x^{1/2}\log x), \tag{9}$$

uniformly in $t\in(-1,0)$. For this purpose we prove that $|H(d,t)|\leqslant 1+\log d$, independently of t. Indeed, by the definition of $h(n,t)$, it is multiplicative and at powers p^j of primes

$$h(p^j,t) = (1+p^{-1}+p^{-2}+\ldots+p^{-j})^t-(1+p^{-1}+\ldots+p^{-j+1})^t.$$

Thus $|h(p^j,t)|\leqslant 1$ and $h(p,t) = (1+1/p)^t-1\geqslant -1/p$ for all $t\in(-1,0)$. Since $h(p,t)<0$, we got that $|h(p,t)|\leqslant 1/p$, and therefore

$$|H(d,t)|\leqslant \sum_{n\leqslant d}\prod_{p|n}|h(p)|\leqslant \sum_{n\leqslant d}\prod_{p|n}1/p < \sum_{n\leqslant d}1/n < 1+\log d.$$

The following estimate now completes the proof of (9).

$$\left|\sum_{n>x}\frac{h(n,t)}{n}\right|\leqslant\left|\sum{}^{*}\sum_{2^j\leqslant n<2^{j+1}}\frac{h(n,t)}{n}\right|\leqslant\sum{}^{*}2^{-j}|H(2^{j+1},t)|$$

$$<\sum{}^{*}\frac{(j+1)\log 2+1}{2^j} = O\left(\frac{\log x}{x}\right),$$

where in $\sum^{*}$ we sum for all integers $j>(\log x)/\log 2$.

After extending the definition of $\sigma(n)$ to all real numbers in an obvious way as a step function, (9) and integration by parts yields that, for $-3/4 \leqslant t < 0$, say,

$$(10) \quad \sum_{n\leqslant x} \sigma^t(n) = \sum_{n\leqslant x} \sigma^t(n)n^{-t}n^t = g(1,t)x^{t+1} + O(x^{1/2+t}\log x) - $$

$$- \int_1^x [g(1,t)y]ty^{t-1}dy + O\left\{\int_1^x (y^{1/2}\log y)ty^{t-1}dy\right\}$$

$$= g(1,t)x^{t+1} - \frac{tg(1,t)}{t+1}x^{t+1} + \frac{tg(1,t)}{t+1} + O(x^{1/2+t}\log x)$$

$$= \frac{g(1,t)}{t+1}x^{t+1} + \frac{tg(1,t)}{t+1} + O(x^{1/2+t}\log x).$$

Observing that

$$(11) \quad \int_{-3/4}^{0} \sum_{2\leqslant n\leqslant x} \sigma^t(n)\,dt = \sum_{2\leqslant n\leqslant x} \int_{-3/4}^{0} \sigma^t(n)\,dt = \sum_{2\leqslant n\leqslant x} \frac{\sigma^t(n)}{\log\sigma(n)}\Bigg|_{t=-3/4}^{0}$$

$$= \sum_{2\leqslant n\leqslant x} \frac{1}{\log\sigma(n)} - \sum_{2\leqslant n\leqslant x} \frac{1}{\sigma^{3/4}(n)\log\sigma(n)},$$

and that by (9)

$$(12) \quad \sum_{2\leqslant n\leqslant x} \frac{1}{\sigma^{3/4}(n)\log\sigma(n)}$$

$$= \sum_{n\leqslant\sqrt{x}} + \sum_{\sqrt{x}\leqslant n\leqslant x} < x^{1/2} + x^{-3/8} \sum_{\sqrt{x}<n\leqslant x} \frac{n^{3/4}}{\sigma^{3/4}(n)\log\sigma(n)} = O(x^{5/8}),$$

we now easily obtain the desired expansion by integrating the right hand side of the outermost equality in (10). Indeed, putting $E(t) = g(1,t)/(t+1)$, repeated integration by parts yields

$$(13) \quad \int_{-3/4}^{0} E(t)x^{t+1}\,dt = x\int_{-3/4}^{0} E(t)x^t\,dt$$

$$= x\frac{E(t)x^t}{\log x}\Bigg|_{t=-3/4}^{0} - x\frac{E'(t)x^t}{(\log x)^2}\Bigg|_{t=-3/4}^{0} + x\frac{E''(t)x^t}{(\log x)^3}\Bigg|_{t=-3/4}^{0} + \ldots$$

$$\ldots + x(-1)^a\frac{E^{(a-1)}(t)x^t}{(\log x)^a}\Bigg|_{t=-3/4}^{0} + \frac{x(-1)^{a+1}}{(\log x)^{a+1}}\int_{-3/4}^{0} E^{(a)}(t)x^t\,dt.$$

Since the integral in the last term of (13) is bounded in x, putting $a_j = (-1)^{j-1}E^{(j-1)}(t)|_{t=0}$, (13) becomes

$$\int_{-3/4}^{0} E(t)x^{t+1}dt = x\sum_{j=1}^{a}\frac{a_j}{(\log x)^j} + O\left(\frac{x}{(\log x)^{a+1}}\right). \tag{14}$$

The integral of the error term in (10) gives

$$x^{1/2}\log x \int_{-3/4}^{0} x^t dt = O(x^{1/2}), \quad \text{and} \quad \int_{-3/4}^{0} tE(t)dt = O(1),$$

which, combined by (10), (11), (12) and (14), completes the proof of the Theorem.

The constructive remarks of the referee are greatly appreciated.

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