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SUMS OF QUOTIENTS OF ADDITIVE FUNCTIONS

JEAN-MARIE DE KONINCK

ABSTRACT. Denote by $\omega(n)$ and $\Omega(n)$ the number of distinct prime factors of n and the total number of prime factors of n, respectively. Given any positive integer α, we prove that

$$\sum_{2\leqq n\leqq x} \Omega(n)/\omega(n) = x + x\sum_{i=1}^{\alpha} a_i/(\log\log x)^i + O(x/(\log\log x)^{\alpha+1}),$$

where $a_1=\sum_p 1/p(p-1)$ and all the other a_i's are computable constants. This improves a previous result of R. L. Duncan.

Denote by $\omega(n)$ and $\Omega(n)$ the number of distinct prime factors of n and the total number of prime factors of n, respectively. R. L. Duncan [3] proved that

$$\sum_{2\leqq n\leqq x} \Omega(n)/\omega(n) = x + O(x/\log\log x).$$

Duncan's result was based on the elementary estimate

$$\sum_{2\leqq n\leqq x} 1/\omega(n) = O(x/\log\log x). \tag{1}$$

In a previous paper [1], we gave estimates of $\sum_{n\leqq x} 1/f(n)$ for a large class of additive functions $f(n)$ (where $\sum'$ denotes summation over those values of n for which $f(n)\neq 0$), which in particular improved considerably the estimate (1). Such sums were further studied by De Koninck and Galambos [2].

In this paper, we prove the following:

THEOREM. *Let α be an arbitrary positive integer; then*

$$\sum_{n\leqq x}{}' \Omega(n)/\omega(n) = x + x\sum_{i=1}^{\alpha} a_i/(\log\log x)^i + O(x/(\log\log x)^{\alpha+1}),$$

where $a_1=\sum_p 1/p(p-1)$ and all the other a_i's are computable constants.

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PROOF. Let t and u be real numbers satisfying $|t|\leqq 1$, $|u|\leqq 1$. Then, for Re $s>1$, we have

$$\sum_{n=1}^{\infty}\frac{t^{\Omega(n)}u^{\omega(n)}}{n^s}=\prod_p\left(1+\frac{tu}{p^s}+\frac{t^2u}{p^{2s}}+\frac{t^3u}{p^{3s}}+\cdots\right)$$
$$=(\zeta(s))^{tu}\prod_p\left(1-\frac{1}{p^s}\right)^{tu}\prod_p\left(1+\frac{tu}{p^s}+\frac{t^2u}{p^{2s}}+\frac{t^3u}{p^{3s}}+\cdots\right)$$
$$=(\zeta(s))^{tu}H(t,u;s),$$

say (Here $\zeta(s)$ denotes the Riemann zeta-function.)

Using a theorem of A. Selberg **[5]**, as we did previously in **[1]**, we obtain that

$$\sum_{n\leqq x}t^{\Omega(n)}u^{\omega(n)}=(H(t,u;1)/\Gamma(tu))x\log^{tu-1}x+O(x\log^{tu-2}x),$$

uniformly for $|t|\leqq 1$, $|u|\leqq 1$, which certainly implies that

$$\sum_{n\leqq x}t^{\Omega(n)}u^{\omega(n)}=\frac{H(t,u;1)}{\Gamma(tu)}x\log^{tu-1}x+O(x/\log x)$$
$$=\frac{x}{\log x}\left\{\frac{H(t,u;1)}{\Gamma(tu)}\log^{tu}x+O(1)\right\}, \tag{2}$$

uniformly for $|t|\leqq 1$, $|u|\leqq 1$.

Now differentiating both sides of (2) with respect to t gives

$$\sum_{n\leqq x}\Omega(n)t^{\Omega(n)-1}u^{\omega(n)}=\frac{x}{\log x}\left\{\log^{tu}x\frac{d}{dt}\left(\frac{H(t,u;1)}{\Gamma(tu)}\right)\right.$$
$$\left.+\frac{H(t,u;1)}{\Gamma(tu)}\cdot\log^{tu}x\cdot\log\log x\cdot u+O(1)\right\},$$

which, by setting $t=1$ and dividing both sides by u, becomes

$$\sum_{n\leqq x}\Omega(n)u^{\omega(n)-1}$$
$$=(x/\log x)\{G(u)\log^u x+F(u)\log^u x\cdot\log\log x+O(1/u)\} \tag{3}$$

uniformly for $|u|\leqq 1$, where

$$G(u)=\frac{1}{u}\frac{d}{dt}\left(\frac{H(t,u;1)}{\Gamma(tu)}\right)\bigg|_{t=1}$$

and

$$F(u)=\frac{H(1,u;1)}{\Gamma(u)}$$

We now proceed to integrate both sides of (3) with respect to u between $\varepsilon(x)=(\log x)^{-1/2}$ and 1 $(x\geqq 3)$. First we have

$$\begin{aligned}\int_{\varepsilon(x)}^{1}\Big(\sum_{2\leqq n\leqq x}\Omega(n)u^{\omega(n)-1}\Big)\,du &= \sum_{2\leqq n\leqq x}\Omega(n)\int_{\varepsilon(x)}^{1}u^{\omega(n)-1}\,du\\ &= \sum_{n\leqq x}{}' \frac{\Omega(n)}{\omega(n)} - \sum_{n\leqq x}{}' \frac{\Omega(n)}{\omega(n)}(\varepsilon(x))^{\omega(n)}\\ &= \sum_{n\leqq x}{}' \frac{\Omega(n)}{\omega(n)} + O\Big(\varepsilon(x)\sum_{2\leqq n\leqq x}\Omega(n)\Big),\end{aligned}$$

since $\omega(n)\geqq 1$ for $n\geqq 2$. It can be proved [4] in an elementary way that $\sum_{2\leqq n\leqq x}\Omega(n)=O(x\log\log x)$. Therefore,

$$\begin{aligned}\int_{\varepsilon(x)}^{1}\Big(\sum_{2\leqq n\leqq x}\Omega(n)u^{\omega(n)-1}\Big)\,du &= \sum_{n\leqq x}{}' \frac{\Omega(n)}{\omega(n)} + O(x(\log\log x)(\log x)^{-1/2})\\ &= \sum_{n\leqq x}{}' \frac{\Omega(n)}{\omega(n)} + O\Big(\frac{x}{(\log\log x)^{\alpha+1}}\Big).\end{aligned}\tag{4}$$

On the other hand, as in [1], repeated integration by parts yields

$$\begin{aligned}&\int_{\varepsilon(x)}^{1}G(u)\log^u x\,du\\ &\quad= \log x\Big\{\frac{G(1)}{\log\log x} - \frac{G'(1)}{(\log\log x)^2} + \frac{G''(1)}{(\log\log x)^3} - \cdots\\ &\qquad\qquad + \frac{(-1)^{\alpha-1}G^{(\alpha-1)}(1)}{(\log\log x)^{\alpha}} + O\Big(\frac{1}{(\log\log x)^{\alpha+1}}\Big)\Big\}\\ &\qquad + \frac{(-1)^{\alpha+1}}{(\log\log x)^{\alpha+1}}\int_{\varepsilon(x)}^{1}G^{(\alpha+1)}(u)\log^u x\,du\\ &\quad= \log x\Big\{\frac{G(1)}{\log\log x} - \frac{G'(1)}{(\log\log x)^2} + \cdots\\ &\qquad\qquad + \frac{(-1)^{\alpha-1}G^{(\alpha-1)}(1)}{(\log\log x)^{\alpha}} + O\Big(\frac{1}{(\log\log x)^{\alpha+1}}\Big)\Big\}.\end{aligned}\tag{5}$$

Similarly we obtain

$$\begin{aligned}&\log\log x\int_{\varepsilon(x)}^{1}F(u)\log^u x\,du\\ &\quad= \log x\Big\{F(1) - \frac{F'(1)}{\log\log x} + \frac{F'(1)}{(\log\log x)^2} - \cdots\\ &\qquad\qquad + \frac{(-1)^{\alpha}F^{(\alpha)}(1)}{(\log\log x)^{\alpha}} + O\Big(\frac{1}{(\log\log x)^{\alpha+1}}\Big)\Big\}.\end{aligned}\tag{6}$$

Finally,

$$\frac{x}{\log x}\int_{\varepsilon(x)}^{1}\frac{du}{u}=O\left(\frac{x\log\varepsilon(x)}{\log x}\right)=O\left(\frac{x\log\log x}{\log x}\right)$$

(7)

$$=O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right)$$

Putting together relations (3), (4), (5), (6) and (7), we have that

$$\sum_{n\leqq x}\frac{\Omega(n)}{\omega(n)}=x\left\{F(1)+\frac{G(1)-F'(1)}{\log\log x}-\frac{G'(1)-F''(1)}{(\log\log x)^2}+\cdots\right.$$
$$\left.+(-1)^{\alpha-1}\frac{G^{(\alpha-1)}(1)-F^{(\alpha)}(1)}{(\log\log x)^{\alpha}}+O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right)\right\}$$

A quite simple computation shows that $F(1)=1$ and that $G(1)-F'(1)=\sum_p 1/p(p-1)$, which proves our Theorem.

From the above reasoning it is clear that similar estimates of $\sum'_{n\leq x} g(n)/f(n)$ could be obtained for a larger class of additive functions f and g along the lines of our previous paper **[1]**.

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Département de Mathématiques, Université Laval, G1K 7P4, Québec, Canada