NOTE ON A FUNCTION SIMILAR TO $n!$

JEAN-MARIE DE KONINCK, Université Laval, Québec

Let $0^{\#} = 1$ and let $n^{\#}$ denote the least common multiple of the integers $1, 2, \cdots, n$. In a recent paper [3], D. Knutson asked if there exists a "Stirling formula" for $n^{\#}$ and furthermore what are the properties of the function $\sum_{n=0}^{\infty} x^n/n^{\#}$. The purpose of this note is to discuss these problems.

The function $n^{\#}$ is well known in number theory. In fact, if we let $\psi(n) = \sum_{p^\alpha \leqq n} \log p$, where the sums runs through all prime powers $\leqq n$, then $\psi(n) = \log n^{\#}$ [2]. Therefore an asymptotic formula for $\log n^{\#}$ can be obtained from the estimate $\psi(n) = n + o(n)$, which is equivalent to the Prime Number Theorem [1].

Concerning the series $\sum_{n=0}^{\infty} x^n/n^{\#}$, we prove that, unlike the series $\sum_{n=0}^{\infty} x^n/n!$ which converges for all real x, it has a finite radius of convergence, namely e.

Indeed, we know that $\sum_{n=0}^{\infty} a_n x^n$ has its radius of convergence equal to $1/\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$. Hence the radius of convergence R of $\sum_{n=0}^{\infty} x^n/n^{\#}$ is equal to

$$\frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{1}{n^{\#}}}} = \frac{1}{\overline{\lim\limits_{n\to\infty}}\, e^{-(1/n)\psi(n)}}.$$

Now $\psi(n) = n + o(n)$ gives

$$\overline{\lim_{n\to\infty}}\, e^{-(1/n)\psi(n)} = \lim_{n\to\infty} e^{-(1/n)\psi(n)} = e^{-1}.$$

Therefore $R = e$.

The author wishes to thank the referee for his suggestions which simplified the presentation of this note.

References

1. R. Ayoub, An Introduction to the Analytic Theory of Numbers, Amer. Math. Soc., Providence, 1963.
2. E. Grosswald, Topics from the Theory of Numbers, Macmillan, New York, 1966.
3. D. Knutson, A lemma on partitions, Amer. Math. Monthly, 79 (1972) 1111–1112.

TRIGONOMETRIC IDENTITIES

ANDY R. MAGID, The University of Oklahoma

A *trigonometric identity* is an equation between two rational functions of trigonometric functions, e.g.:

$$\frac{\tan x}{\csc x - \cot x} - \frac{\sin x}{\csc x + \cot x} = \sec x + \cos x.$$