ON A CLASS OF ARITHMETICAL FUNCTIONS

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1. Introduction. Let $\omega(n)$ be the number of distinct prime divisors of n. Then estimates for $\sum_{n\le x}\omega(n)$ are well known [3]. On the other hand, estimates for $\sum'_{n\le x} 1/\omega(n)$ were only recently studied [1], [2]. (From here on, the prime in a sum of the form $\sum'_{n\le x} 1/f(n)$ means that the sum is taken over all $n \le x$ such that $f(n) \ne 0$.)

Using Turan's inequality, R. L. Duncan proves in [1] that

$$\sum_{n\le x}' \frac{1}{\omega(n)} = O\left(\frac{x}{\log\log x}\right)$$

and then uses this result to show that $\Omega(n)/\omega(n)$ has average order one, where $\Omega(n)$ stands for the total number of prime divisors of n.

In this paper, we obtain a much better estimate for $\sum'_{n\le x} 1/\omega(n)$ and we also obtain estimates for $\sum'_{n\le x} 1/(f(n))^k$ for a large class of arithmetical functions $\{f(n)\}$ and an arbitrary positive integer k.

2. A result of A. Selberg and basic definitions. Before defining our class of functions, we state a result of A. Selberg [4]. Restricted to the particular case needed here the result may be stated as follows.

THEOREM A (Selberg). *Let $g(s, t) = \sum_{n=1}^{\infty} b_t(n)/n^s$ for* Re $s = \sigma > 1$, *and let $\sum_{n=1}^{\infty} |b_t(n)|\, n^{-1} \log^{B+3} 2n$ be uniformly bounded for $|t| \le B$. Next, set $(\zeta(s))^t g(s, t) = \sum_{n=1}^{\infty} a_t(n)/n^s$ for $\sigma > 1$. Then we have $\sum_{n\le x} a_t(n) = (g(1, t)/\Gamma(t))\, x \log^{t-1} x + O\,(x \log^{t-2} x)$ uniformly for $|t| \le B$, $x \ge 2$. (Here $\zeta(s)$ stands for the Riemann zeta function.)*

DEFINITION 1. Let S denote the set of all real-valued arithmetical functions satisfying the following two conditions.

(1) $f(n) \ne 0 \Rightarrow f(n) \ge 1$ for each integer $n \ge 1$.

(2) $\displaystyle\sum_{\substack{n\le x\\ f(n)=0}} 1 = O\left(\frac{x}{\log x}\right)$.

DEFINITION 2. Given α (from now on, unless otherwise mentioned, α stands for an arbitrary positive integer), we denote by S_α the set of those functions in S for which $t^{f(n)} = a_t(n)$ satisfies the conditions of Theorem A, with $B = 1$ and $D(t) = (g(1, t)/\Gamma(t)) \in C^{\alpha+1}[0, 1]$.

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Notation. If it exists, the i-th derivative of $h(t)$ with respect to t will be denoted by $h^{(i)}(t)$ or $(h(t))^{(i)}$.

Definition 3. Given a function $f \in S_\alpha$, let $D(t)$ be the corresponding function of Definition 2; then for $t \in (0, 1]$ set

$$B_i(t) = \left(\frac{D(t)}{t}\right)^{(i-1)}$$

and $A_i(t) = (-1)^{i-1}B_i(t)$ for $i = 1, 2, \cdots, \alpha + 2$. Sometimes we shall write A_i for $A_i(1)$.

3. Main theorem concerning estimates of $\sum'_{n\leq x} 1/f(n)$. We first prove two lemmas which we will need in the proof of our main theorem (Theorem 3).

Lemma 1. *Given $f \in S_\alpha$, with corresponding functions $A_i(t)$, $|A_i(t)| \leq M/t^{\alpha+2}$ holds uniformly for $t \in (0, 1]$ and $1 \leq i \leq \alpha + 2$, with some constant M depending only on f.*

Proof. The proof is immediate from Definition 3.

Lemma 2. *Let $\frac{1}{2} < \eta \leq 1$. Let $\epsilon(x) = (\log x)^{-(1/2(\alpha+2))}$ for $x \geq 3$. Then if x is sufficiently large,*

$$\operatorname*{Max}_{\epsilon(x)\leq t\leq\eta} \frac{\log^t x}{t^{\alpha+2}} = \frac{\log^\eta x}{\eta^{\alpha+2}}.$$

Proof. Let $h(t) = (\log^t x/t^{\alpha+2})$ and suppose that x is large enough so that $\epsilon(x) \leq \eta \leq 1$. Then

$$h'(t) = \frac{\log^t x}{t^{\alpha+3}}(t \log\log x - \alpha - 2).$$

Setting $h'(t) = 0$, we get $t = (\alpha + 2)/\log\log x$. On the other hand,

$$h''(t) = \frac{\log^t x}{t^{\alpha+4}}\{t \log\log x + (t \log\log x - \alpha - 2)(t \log\log x - \alpha - 3)\},$$

which is strictly positive at $t = (\alpha + 2)/\log\log x$. Therefore $h(t)$ has a minimum at $t = (\alpha + 2)/\log\log x$ and there are no other local maxima or minima on $[\epsilon(x), \eta]$. So $h(t)$ is decreasing between $\epsilon(x)$ and $(\alpha + 2)/\log\log x$ and is increasing between $(\alpha + 2)/\log\log x$ and η if x is sufficiently large. Therefore

$$\operatorname*{Max}_{\epsilon(x)\leq t\leq\eta} \frac{\log^t x}{t^{\alpha+2}} = \operatorname{Max}\left(\frac{\log^{\epsilon(x)} x}{(\epsilon(x))^{\alpha+2}}, \frac{\log^\eta x}{\eta^{\alpha+2}}\right).$$

But because $\eta > \frac{1}{2}$

$$\frac{\log^{\epsilon(x)} x}{(\epsilon(x))^{\alpha+2}} = (\log^{\epsilon(x)} x)(\log^{\frac{1}{2}} x) < \log^\eta x$$

for x sufficiently large. And obviously, since $\eta \leq 1$,

$$\log^\eta x \leq \frac{\log^\eta x}{\eta^{\alpha+2}},$$

whence the lemma follows.

We are now ready to prove our main result.

THEOREM 3. *Let $f \varepsilon S_\alpha$; then*

$$\sum_{n\leq x}' \frac{1}{f(n)} = x \sum_{i=1}^{\alpha} \frac{A_i}{(\log \log x)^i} + O\left(\frac{x}{(\log \log x)^{\alpha+1}}\right).$$

Proof. Since $f \varepsilon S_\alpha$, we have

$$\sum_{n\leq x} t^{f(n)} = D(t)x \log^{t-1} x + R(x, t)$$

with $R(x, t) = O\ (x \log^{t-2} x)$ uniformly for $t \varepsilon [0, 1]$ and $D(t) \varepsilon C^{\alpha+1}[0, 1]$.

Now since $f \varepsilon S$, we can write

$$(1) \qquad \sum_{\substack{n\leq x\\ f(n)\neq 0}} t^{f(n)} = \sum_{n\leq x} t^{f(n)} - \sum_{\substack{n\leq x\\ f(n)=0}} t^{f(n)} = D(t)x \log^{t-1} x + R(x, t) + R_1(x),$$

with $R_1(x) = O\ (x/\log x)$. Dividing by t and recalling Definition 3, (1) becomes

$$(2) \qquad \sum_{\substack{n\leq x\\ f(n)\neq 0}} t^{f(n)-1} = B_1(t)x \log^{t-1} x + \frac{R(x, t)}{t} + \frac{R_1(x)}{t}.$$

Let $\epsilon(x) = (\log x)^{-(1/2(\alpha+2))}$, as in Lemma 2, and suppose that $x \geq 3$ (so that $\epsilon(x) < 1$); then

$$(3) \qquad \int_{\epsilon(x)}^1 \left(\sum_{\substack{n\leq x\\ f(n)\neq 0}} t^{f(n)-1}\right) dt$$
$$= \int_{\epsilon(x)}^1 B_1(t)x \log^{t-1} x\, dt + \int_{\epsilon(x)}^1 \frac{R(x, t)}{t}\, dt + R_1(x) \int_{\epsilon(x)}^1 \frac{dt}{t}.$$

One can easily show that the last two terms on the right of (3) are $O\ (x/(\log \log x)^{\alpha+1})$. On the other hand,

$$(4) \qquad \int_{\epsilon(x)}^1 \left(\sum_{\substack{n\leq x\\ f(n)\neq 0}} t^{f(n)-1}\right) dt = \sum_{n\leq x}' \frac{1}{f(n)} - \sum_{n\leq x}' \frac{(\epsilon(x))^{f(n)}}{f(n)}.$$

But since $f \varepsilon S$ and $0 < \epsilon(x) < 1$, the last term on the right of (4) is also $O\ (x/(\log \log x)^{\alpha+1})$. Therefore, using (3) and (4), we have

$$(5) \qquad \sum_{n\leq x}' \frac{1}{f(n)} = \int_{\epsilon(x)}^1 B_1(t)x \log^{t-1} x\, dt + O\left(\frac{x}{(\log \log x)^{\alpha+1}}\right).$$

Integrating by parts and recalling Definition 3 yields

$$\text{(6)}\qquad \int_{\epsilon(x)}^{1} B_1(t)x \log^{t-1} x\, dt = x\left\{ \sum_{i=1}^{\alpha} \frac{A_i(t)\log^{t-1} x}{(\log\log x)^i}\Bigg|_{\epsilon(x)}^{1} + \frac{A_{\alpha+1}(t)\log^{t-1} x}{(\log\log x)^{\alpha+1}}\Bigg|_{\epsilon(x)}^{1} + \frac{1}{(\log\log x)^{\alpha+1}}\int_{\epsilon(x)}^{1} A_{\alpha+2}(t)\log^{t-1} x\, dt\right\}.$$

Using Lemma 1, we see that for $1 \leq i \leq \alpha + 1$

$$\text{(7)}\qquad \left|\frac{A_i(\epsilon(x))\log^{\epsilon(x)-1} x}{(\log\log x)^i}\right| \leq \frac{M \log^{\epsilon(x)-1} x}{(\epsilon(x))^{\alpha+2}(\log\log x)^i} = O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right).$$

On the other hand, from Lemma 1 and Lemma 2, we have

$$\text{(8)}\qquad \left|\int_{\epsilon(x)}^{1} A_{\alpha+2}(t)\log^{t-1} x\, dt\right| < \int_{\epsilon(x)}^{1} |A_{\alpha+2}(t)| \log^{t-1} x\, dt < M \cdot \operatorname*{Max}_{\epsilon(x)\leq t\leq 1} \frac{\log^{t-1} x}{t^{\alpha+2}} = \frac{M}{\log x}\log x = M.$$

Finally, observing that $A_{\alpha+1}(1) = O(1)$ and using (7) and (8), we find that (6) can be written

$$\text{(9)}\qquad \int_{\epsilon(x)}^{1} B_1(t)x\log^{t-1} x\, dt = x\left\{\sum_{i=1}^{\alpha}\frac{A_i}{(\log\log x)^i} + O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right)\right\}.$$

Putting (5) and (9) together gives our theorem.

Observing that $\omega(n)$ and $\Omega(n)$ belong to S_α for any α, we obtain from Theorem 3, after a simple computation, the following applications that we state as theorems.

THEOREM 4.

$$\sum_{n\leq x}{}' \frac{1}{\omega(n)} = x\sum_{i=1}^{\alpha}\frac{a_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where $a_1 = 1$, $a_2 = 1 - \rho$, *with*

$$\rho = \gamma + \sum_p \{\log(1 - p^{-1}) + p^{-1}\},$$

and all the other a_i*'s are computable constants.* (*Here* γ *stands for the Euler constant.*)

THEOREM 5.

$$\sum_{n\leq x}{}' \frac{1}{\Omega(n)} = x\sum_{i=1}^{\alpha}\frac{b_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where $b_1 = 1, b_2 = 1 - \rho - \sum_p 1/p(p-1)$, *and all the other* b_i*'s are computable constants.*

Professor D. Rearick in a private communication mentioned that it would be interesting if one could use our method to estimate $\sum'_{n \le x} 1/\log d(n)$, where $d(n)$ denotes the number of divisors of n. We can prove the following.

THEOREM 6.

$$\sum_{n \le x}' \frac{1}{\log d(n)} = x \sum_{i=1}^{\alpha} \frac{c_i}{(\log \log x)^i} + O\left(\frac{x}{(\log \log x)^{\alpha+1}}\right),$$

where

$$c_1 = 1/\log 2, \qquad c_2 = \frac{1}{\log 2}\left(1 - \rho - \frac{1}{\log 2} \sum_p \left(\frac{\log \frac{2}{3}}{p^2} + \frac{\log \frac{3}{4}}{p^3} + \cdots\right)\right),$$

and all the other c_i*'s are computable constants.* (*Here* ρ *is the constant defined in Theorem* 4.)

Proof. We have for $\sigma > 1$ and $t \in [0, 1]$

$$\sum_{n=1}^{\infty} \frac{t^{\log d(n)}}{n^s} = \prod_p \left(1 + \frac{t^{\log 2}}{p^s} + \frac{t^{\log 3}}{p^{2s}} + \cdots\right).$$

Let $u = t^{\log 2}$ so that $t = u^{1/\log 2}$ and

$$\sum_{n=1}^{\infty} \frac{u^{\log d(n)/\log 2}}{n^s} = \prod_p \left(1 + \frac{u}{p^s} + \frac{u^{\log 3/\log 2}}{p^{2s}} + \cdots\right)$$
$$= (\zeta(s))^u \prod_p (1 - p^{-s})^u \prod_p (\cdots)$$
$$= (\zeta(s))^u g(s, u).$$

We can easily see that $\log d(n)/\log 2$ belongs to S_α for any α and that $D(u) = (g(1, u)/\Gamma(u))$, where $g(1, u) = e^{w(u)}$ with

$$w(u) = \sum_p u \log (1 - p^{-1}) + \log\left(1 + \frac{u}{p} + \frac{u^{\log 3/\log 2}}{p^2} + \frac{u^{\log 4/\log 2}}{p^3} + \cdots\right).$$

And Theorem 3 gives us

$$\sum_{n \le x}' \frac{1}{\frac{\log d(n)}{\log 2}} = x \sum_{i=1}^{\alpha} \frac{A_i}{(\log \log x)^i} + O\left(\frac{x}{(\log \log x)^{\alpha+1}}\right),$$

where

$$A_1 = 1, \qquad A_2 = 1 - \rho - \frac{1}{\log 2} \sum_p \left(\frac{\log \frac{2}{3}}{p^2} + \frac{\log \frac{3}{4}}{p^3} + \cdots\right),$$

and all the other A_i's are computable constants, that is,

$$\sum_{n\le x}{}' \frac{1}{\log d(n)} = x \sum_{i=1}^{\alpha} \frac{c_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

with the desired c_i's.

4. A generalization of the main theorem.

DEFINITION 4. Let S_α^* be the set of all ordered pairs of arithmetical functions (g, f) which satisfy the following four conditions.

(1) $f \varepsilon S$.

(2) $\displaystyle\sum_{\substack{n\le x\\ f(n)\ne 0}} g(n) = O\left(\frac{x}{\log x}\right)$.

(3) $\displaystyle\sum_{\substack{n\le x\\ f(n)\ne 0}} g(n) = O(x)$.

(4) $g(n)t^{f(n)} = a_t(n)$ satisfies the conditions of Theorem A, with $B = 1$ and $D(t) = (g(1, t)/\Gamma(t)) \varepsilon C^{\alpha+1}[0, 1]$.

From this definition, we observe that if $(g, f) \varepsilon S_\alpha^*$, then

$$\sum_{n\le x} g(n)t^{f(n)} = D(t)x \log^{t-1} x + O(x \log^{t-2} x)$$

uniformly for $|t| \le 1$. Therefore to each ordered pair $(g, f) \varepsilon S_\alpha^*$ we can associate the function $D(t)$, and using this definition, we define the functions $B_i(t)$ and $A_i(t)$, $1 \le i \le \alpha + 2$, as in Definition 3. We can now state the following theorem.

THEOREM 7. *Let $(g, f) \varepsilon S_\alpha^*$; then*

$$\sum_{n\le x}{}' \frac{g(n)}{f(n)} = x \sum_{i=1}^{\alpha} \frac{A_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right).$$

Proof. Taking into account Definition 4, we see that the proof is entirely similar to the one of Theorem 3.

Observing that $(\mu^2, \omega) \varepsilon S_\alpha^*$, where μ stands for the Möbius function, the following application follows from Theorem 7.

THEOREM 8.

$$\sum_{n\le x}{}' \frac{\mu^2(n)}{\omega(n)} = x \sum_{i=1}^{\alpha} \frac{d_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where

$$d_1 = \frac{6}{\pi^2}, \qquad d_2 = \frac{6}{\pi^2}\left(1 - \rho + \sum_p \frac{1}{p(p+1)}\right),$$

and all the other d_i's are computable constants.

5. Estimates for $\sum'_{n\le x} 1/(f(n))^k$ for an arbitrary positive integer k. In this section we obtain estimates for $\sum'_{n\le x} 1/(f(n))^k$ for functions $f \in S_\alpha$, with k a positive integer, $k \le \alpha$. We first make a definition and state two lemmas that will be used in the next theorem.

DEFINITION 5. For $t \in (0, 1]$ set

$$ {}^1A_i(t) = A_i(t) \quad \text{and} \quad {}^1B_i(t) = B_i(t) $$

for $i = 1, 2, \cdots, \alpha + 2$, with $A_i(t)$ and $B_i(t)$ as in Definition 3. Next, define

$$ {}^2B_i(t) = \sum_{j=1}^{i-1} \left(\frac{{}^1B_j(t)}{t}\right)^{(i-j-1)} $$

and

$$ {}^2A_i(t) = (-1)^{i-2}\, {}^2B_i(t) $$

for $i = 2, 3, \cdots, \alpha + 2$. More generally, for $2 \le k \le \alpha$ set

$$ {}^kB_i(t) = \sum_{j=k-1}^{i-1} \left(\frac{{}^{k-1}B_j(t)}{t}\right)^{(i-j-1)} $$

and

$$ {}^kA_i(t) = (-1)^{i-k}\, {}^kB_i(t) $$

for $i = k, k + 1, \cdots, \alpha + 2$. We will also write ${}^kA_i$ for ${}^kA_i(1)$.

LEMMA 9. *Let $f \in S_\alpha$ and let k be a positive integer, $k \le \alpha$. To f associate the corresponding functions ${}^kA_i(t)$, $k \le i \le \alpha + 2$. Then*
(1) *there exists a constant M, depending only on f, such that $|{}^kA_i(t)| \le M/t^i \le M/t^{\alpha+2}$ uniformly for $t \in (0, 1]$ and $k \le i \le \alpha + 2$, and*
(2) *there exists a constant N, depending only on f, such that $|({}^kA_i(t)/t)'| \le N/t^{\alpha+2}$ uniformly for $t \in (0, 1]$ and $k \le i \le \alpha$.*

Proof. This lemma is simply a generalization of Lemma 1 and it follows immediately from Definition 5.

LEMMA 10. *Let $x \ge 3$ and $\epsilon(x) \le u \le 1$. Let B be a positive integer. Then*

$$ \operatorname*{Max}_{\epsilon(x)\le t\le u} \frac{\log^t x}{t^B} \le \frac{\log^{\epsilon(x)} x}{(\epsilon(x))^B} + \frac{\log^u x}{u^B}. $$

Proof. Let $h(t) = (\log^t x)/t^B$. From the proof of Lemma 2, it is easily seen that the only two possible maxima of $h(t)$ in $[\epsilon(x), u]$ are $\epsilon(x)$ and u. And our lemma follows from this observation.

We now establish a general formula which will help us find our estimate for $\sum'_{n\le x} 1/(f(n))^k$.

THEOREM 11. *Let $f \in S_\alpha$ and $x \ge 3$. Let k be an arbitrary positive integer, $k \le \alpha$. Then*

$$(10)\qquad \sum_{n\le x}{}' \frac{u^{f(n)}}{(f(n))^k} = x \sum_{i=k}^{\alpha} \frac{{}^k A_i(u) \log^{u-1} x}{(\log\log x)^i} + O\left(\frac{x \log^{u-1} x}{u^{\alpha+k+1}(\log\log x)^{\alpha+1}}\right) + O(x(\log\log x)^{k-1}\epsilon(x))$$

uniformly for $u \in [\epsilon(x), 1]$.

Proof. The proof is by induction on k. Since $f \in S_\alpha$, Equation (2) holds and we have for $\epsilon(x) \le v \le 1$

$$(11)\qquad \int_{\epsilon(x)}^{v} \Big(\sum_{\substack{n\le x\\ f(n)\ne 0}} t^{f(n)-1}\Big)\, dt = \int_{\epsilon(x)}^{v} B_1(t) x \log^{t-1} x\, dt + \int_{\epsilon(x)}^{v} \frac{R(x,t)}{t}\, dt + R_1(x) \int_{\epsilon(x)}^{v} \frac{dt}{t}.$$

As before, the last two terms on the right of (11) are easily shown to be $O(x\epsilon(x))$. Now

$$\int_{\epsilon(x)}^{v} \Big(\sum_{\substack{n\le x\\ f(n)\ne 0}} t^{f(n)-1}\Big)\, dt = \sum_{n\le x}{}' \frac{v^{f(n)}}{f(n)} - \sum_{n\le x}{}' \frac{(\epsilon(x))^{f(n)}}{f(n)} = \sum_{n\le x}{}' \frac{v^{f(n)}}{f(n)} + O(x\epsilon(x)).$$

Hence (11) becomes

$$(12)\qquad \sum_{n\le x}{}' \frac{v^{f(n)}}{f(n)} = \int_{\epsilon(x)}^{v} B_1(t) x \log^{t-1} x\, dt + O(x\epsilon(x)).$$

By repeated integration by parts as in the proof of Theorem 3 and recalling Definition 3, we see that

$$(13)\qquad \int_{\epsilon(x)}^{v} B_1(t) x \log^{t-1} x\, dt = x\left\{ \sum_{i=1}^{\alpha} \frac{A_i(t) \log^{t-1} x}{(\log\log x)^i} \Bigg|_{\epsilon(x)}^{v} + \frac{A_{\alpha+1}(t)\log^{t-1} x}{(\log\log x)^{\alpha+1}} \Bigg|_{\epsilon(x)}^{v} + \frac{1}{(\log\log x)^{\alpha+1}} \int_{\epsilon(x)}^{v} A_{\alpha+2}(t) \log^{t-1} x\, dt \right\}.$$

The following two estimates hold on account of Lemma 1.

$$\frac{A_i(\epsilon(x)) \log^{\epsilon(x)-1} x}{(\log\log x)^i} = O(\epsilon(x)), \qquad \text{for } 1 \le i \le \alpha + 1,$$

$$\frac{A_{\alpha+1}(v) \log^{v-1} x}{(\log\log x)^{\alpha+1}} = O\left(\frac{\log^{v-1} x}{v^{\alpha+2}(\log\log x)^{\alpha+1}}\right)$$

And using Lemma 1 and Lemma 10, we also have

$$\left|\int_{\epsilon(x)}^{v} A_{\alpha+2}(t) \log^{t-1} x \, dt\right| < \int_{\epsilon(x)}^{v} |A_{\alpha+2}(t)| \log^{t-1} x \, dt$$

$$< \frac{M}{\log x} \operatorname*{Max}_{\epsilon(x) \le t \le v} \frac{\log^t x}{t^{\alpha+2}} = O\left(\frac{\log^{v-1} x}{v^{\alpha+2}}\right) + O(\epsilon(x));$$

therefore

$$\frac{1}{(\log \log x)^{\alpha+1}} \int_{\epsilon(x)}^{v} A_{\alpha+2}(t) \log^{t-1} x \, dt = O\left(\frac{\log^{v-1} x}{v^{\alpha+2}(\log \log x)^{\alpha+1}}\right) + O(\epsilon(x)).$$

By using these estimates (13) becomes

$$\int_{\epsilon(x)}^{v} B_1(t) x \log^{t-1} x \, dt = x \sum_{i=1}^{\alpha} \frac{A_i(v) \log^{v-1} x}{(\log \log x)^i} + O\left(\frac{x \log^{v-1} x}{v^{\alpha+2}(\log \log x)^{\alpha+1}}\right) + O(x\epsilon(x))$$

uniformly for $v \in [\epsilon(x), 1]$. Using this relation and Equation (12), we obtain

$$\sum_{n \le x}{}' \frac{v^{f(n)}}{f(n)} = x \sum_{i=1}^{\alpha} \frac{A_i(v) \log^{v-1} x}{(\log \log x)^i} + O\left(\frac{x \log^{v-1} x}{v^{\alpha+2}(\log \log x)^{\alpha+1}}\right) + O(x\epsilon(x))$$

uniformly for $v \in [\epsilon(x), 1]$. Therefore Formula (10) holds for $k = 1$. The proof of (10) is now completed by induction on k.

Let us assume that (10) holds for $k = m$, $m < \alpha$. By the induction hypothesis we have

$$\sum_{n \le x}{}' \frac{v^{f(n)}}{(f(n))^m} = x \sum_{i=m}^{\alpha} \frac{{}^m A_i(v) \log^{v-1} x}{(\log \log x)^i} + W(x, v) + W_1(x) \tag{14}$$

with

$$W(x, v) = O\left(\frac{x \log^{v-1} x}{v^{\alpha+m+1}(\log \log x)^{\alpha+1}}\right)$$

uniformly for $v \in [\epsilon(x), 1]$ and $W_1(x) = O\ (x(\log \log x)^{m-1}\epsilon(x))$. Now dividing Equation (14) by v, integrating both sides between $\epsilon(x)$ and u, $\epsilon(x) \le u \le 1$, and using Definition 5 and Lemma 10 as before, we obtain

$$\sum_{n \le x}{}' \frac{u^{f(n)}}{(f(n))^{m+1}} = x \sum_{i=m}^{\alpha} \frac{1}{(\log \log x)^i} \int_{\epsilon(x)}^{u} \frac{{}^m A_i(v)}{v} \log^{v-1} x \, dv$$

$$+ O\left(\frac{x \log^{u-1} x}{u^{\alpha+m+2}(\log \log x)^{\alpha+1}}\right) + O(x(\log \log x)^m \epsilon(x)).$$

Our theorem will be proved if we can show that

$$\sum_{i=m}^{\alpha} \frac{1}{(\log \log x)^i} \int_{\epsilon(x)}^{u} \frac{{}^m A_i(v)}{v} \log^{v-1} x \, dv \tag{15}$$

$$= \sum_{r=m+1}^{\alpha} \frac{{}^{m+1}A_r(u) \log^{u-1} x}{(\log \log x)^r} + O\left(\frac{\log^{u-1} x}{u^{\alpha+2}(\log \log x)^{\alpha+1}}\right) + O(\epsilon(x)).$$

Now observe that the last term in the sum on the left side of relation (15) is

$$\frac{1}{(\log\log x)^{\alpha}}\int_{\epsilon(x)}^{u}\frac{{}^{m}A_{\alpha}(v)}{v}\log^{v-1}x\,dv.$$

Using integration by parts and Lemma 9 and Lemma 10, we see that it can be shown that this term is

$$O\left(\frac{\log^{u-1}x}{u^{\alpha+2}(\log\log x)^{\alpha+1}}\right)+O(\epsilon(x)).$$

From this and relation (15), we observe that our theorem will be proved if we can show that

$$(15')\qquad \sum_{i=m}^{\alpha-1}\frac{1}{(\log\log x)^{i}}\int_{\epsilon(x)}^{u}\frac{{}^{m}A_{i}(v)}{v}\log^{v-1}x\,dv$$
$$=\sum_{r=m+1}^{\alpha}\frac{{}^{m+1}A_{r}(u)\log^{u-1}x}{(\log\log x)^{r}}+O\left(\frac{\log^{u-1}x}{u^{\alpha+2}(\log\log x)^{\alpha+1}}\right)+O(\epsilon(x)).$$

Integrating by parts for each $m\le i<\alpha$, as in the proof of Theorem 3, we obtain

$$\sum_{i=m}^{\alpha-1}\frac{1}{(\log\log x)^{i}}\int_{\epsilon(x)}^{u}\frac{{}^{m}A_{i}(v)}{v}\log^{v-1}x\,dv$$
$$=\sum_{i=m}^{\alpha-1}\sum_{j=1}^{\alpha-i}(-1)^{j-1}\left(\frac{{}^{m}A_{i}(v)}{v}\right)^{(j-1)}\frac{\log^{v-1}x}{(\log\log x)^{i+j}}\Bigg|_{\epsilon(x)}^{u}$$
$$+\sum_{i=m}^{\alpha-1}(-1)^{\alpha-i}\left(\frac{{}^{m}A_{i}(v)}{v}\right)^{(\alpha-i)}\frac{\log^{v-1}x}{(\log\log x)^{\alpha+1}}\Bigg|_{\epsilon(x)}^{u}$$
$$+\sum_{i=m}^{\alpha-1}\frac{(-1)^{\alpha-i+1}}{(\log\log x)^{\alpha+1}}\int_{\epsilon(x)}^{u}\left(\frac{{}^{m}A_{i}(v)}{v}\right)^{(\alpha-i+1)}\log^{v-1}x\,dv$$
$$=I_1+I_2+I_3,$$

say. We now estimate separately I_1, I_2 and I_3.

$$I_1=\sum_{i=m}^{\alpha-1}\sum_{j=1}^{\alpha-i}\left(\frac{{}^{m}B_{i}(v)}{v}\right)^{(j-1)}\frac{(-1)^{i+j-m-1}\log^{v-1}x}{(\log\log x)^{i+j}}\Bigg|_{\epsilon(x)}^{u}$$
$$=\sum_{r=m+1}^{\alpha}\frac{(-1)^{r-m-1}\sum_{l=m}^{r-1}\left(\frac{{}^{m}B_{l}(v)}{v}\right)^{(r-l-1)}\log^{v-1}x}{(\log\log x)^{r}}\Bigg|_{\epsilon(x)}^{u}$$

by Definition 5, and this equals

$$\sum_{r=m+1}^{\alpha}\frac{(-1)^{r-m-1}\,{}^{m+1}B_{r}(v)\log^{v-1}x}{(\log\log x)^{r}}\Bigg|_{\epsilon(x)}^{u}=\sum_{r=m+1}^{\alpha}\frac{{}^{m+1}A_{r}(v)\log^{v-1}x}{(\log\log x)^{r}}\Bigg|_{\epsilon(x)}^{u}$$
$$=\sum_{r=m+1}^{\alpha}\frac{{}^{m+1}A_{r}(u)\log^{u-1}x}{(\log\log x)^{r}}+O(\epsilon(x))$$

by Lemma 9.

On the other hand, recalling the definition of ${}^{m+1}B_{\alpha+1}(v)$, we have

$$I_2 = \frac{\log^{v-1} x}{(\log \log x)^{\alpha+1}} \sum_{i=m}^{\alpha-1} (-1)^{\alpha-i}(-1)^{i-m}\left(\frac{{}^mB_i(v)}{v}\right)^{(\alpha-i)} \Bigg|_{\epsilon(x)}^{u}$$

$$= \frac{\log^{v-1} x}{(\log \log x)^{\alpha+1}} (-1)^{\alpha-m} \sum_{i=m}^{\alpha-1} \left(\frac{{}^mB_i(v)}{v}\right)^{(\alpha-i)} \Bigg|_{\epsilon(x)}^{u}$$

$$= \frac{\log^{v-1} x}{(\log \log x)^{\alpha+1}} (-1)^{\alpha-m}\left({}^{m+1}B_{\alpha+1}(v) - \frac{{}^mB_\alpha(v)}{v}\right) \Bigg|_{\epsilon(x)}^{u};$$

using Lemma 9, we see that I_2 is

$$O\left(\frac{\log^{u-1} x}{u^{\alpha+2}(\log \log x)^{\alpha+1}}\right) + O(\epsilon(x)).$$

Finally, recalling the definition of ${}^{m+1}B_{\alpha+2}(v)$, we have

$$I_3 = \frac{1}{(\log \log x)^{\alpha+1}} \int_{\epsilon(x)}^{u} (-1)^{\alpha-m+1} \sum_{i=m}^{\alpha-1} \left(\frac{{}^mB_i(v)}{v}\right)^{(\alpha-i+1)} \log^{v-1} x \, dv$$

$$= \frac{1}{(\log \log x)^{\alpha+1}}$$

$$\cdot \int_{\epsilon(x)}^{u} (-1)^{\alpha-m+1}\left({}^{m+1}B_{\alpha+2}(v) - \left(\frac{{}^mB_\alpha(v)}{v}\right)' + \frac{{}^mB_{\alpha+1}(v)}{v}\right) \log^{v-1} x \, dv;$$

again by Lemma 9 we see that I_3 is

$$O\left(\frac{\log^{u-1} x}{u^{\alpha+2}(\log \log x)^{\alpha+1}}\right) + O(\epsilon(x)).$$

Putting together these estimates, (15′) follows and the theorem is proved.

From Theorem 11 we easily obtain the final desired result.

THEOREM 12. *Let $f \in S_\alpha$; then for an arbitrary positive integer $k \leq \alpha$*

$$\sum_{n \leq x}' \frac{1}{(f(n))^k} = x \sum_{i=k}^{\alpha} \frac{{}^kA_i}{(\log \log x)^i} + O\left(\frac{x}{(\log \log x)^{\alpha+1}}\right).$$

Proof. The proof is immediate from Theorem 11 by substituting $u = 1$ in (10) and by observing that

$$x(\log \log x)^{k-1}\epsilon(x) = O\left(\frac{x}{(\log \log x)^{\alpha+1}}\right).$$

We now indicate three applications which follow essentially from Theorem 12 and Definition 5.

THEOREM 13. *Let $\alpha \geq 2$; then*

$$\sum_{n\le x}{}' \frac{1}{\omega^2(n)} = x \sum_{i=2}^{\alpha} \frac{e_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where $e_2 = 1$, $e_3 = 3 - 2\rho$, *and all the other* e_i*'s are computable constants.*

THEOREM 14. *Let* $\alpha \ge 2$; *then*

$$\sum_{n\le x}{}' \frac{1}{\Omega^2(n)} = x \sum_{i=2}^{\alpha} \frac{m_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where $m_2 = 1$, $m_3 = 3 - 2\rho - 2\sum_p 1/p(p-1)$, *and all the other* m_i*'s are computable constants.*

THEOREM 15. *Let* $\alpha \ge 2$; *then*

$$\sum_{n\le x}{}' \frac{1}{\log^2 d(n)} = x \sum_{i=2}^{\alpha} \frac{q_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where

$$q_2 = \frac{1}{\log^2 2}, \qquad q_3 = \frac{1}{\log^2 2}\left(3 - 2\rho - \frac{2}{\log 2}\sum_p \left(\frac{\log \frac{2}{3}}{p^2} + \frac{\log \frac{3}{4}}{p^3} + \cdots\right)\right),$$

and all the other q_i*'s are computable constants.*

Finally from an obvious generalization of Theorem 7 and Theorem 12, the next theorem follows immediately.

THEOREM 16. *Let* $\alpha \ge 2$; *then*

$$\sum_{n\le x}{}' \frac{\mu^2(n)}{\omega^2(n)} = x \sum_{i=2}^{\alpha} \frac{r_i}{(\log\log x)^i} + O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right),$$

where

$$r_2 = \frac{6}{\pi^2}, \qquad r_3 = \frac{6}{\pi^2}\left(3 - 2\rho + 2\sum_p \frac{1}{p(p+1)}\right),$$

and all the other r_i*'s are computable constants.*

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