On the $k$-fold iterate of the sum of divisors function

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Abstract

Let $\gamma(n)$ stand for the product of the prime factors of $n$. The index of composition $\lambda(n)$ of an integer $n \geq 2$ is defined as $\lambda(n) = \log n / \log \gamma(n)$ with $\lambda(1) = 1$. Given an arbitrary integer $k \geq 0$ and letting $\sigma_k(n)$ be the $k$-fold iterate of the sum of divisors function, we show that, given any real number $\varepsilon > 0$, $\lambda(\sigma_k(n)) < 1 + \varepsilon$ for almost all positive integers $n$.

1 Introduction and notation

Let $\gamma(n)$ stand for the product of the prime factors of the positive integer $n$. In 2003, De Koninck and Doyon [DD] studied the mean value and various
other properties of the index of composition of an integer, defined for \( n \geq 2 \) by \( \lambda(n) := \frac{\log n}{\log \gamma(n)} \), with \( \lambda(1) = 1 \). Later, others (see [DK], [DKS], [ZZ]) further studied the behaviour of this function. Of particular interest is the result of De Koninck and Luca [DL] who showed that the normal order of \( \lambda(\sigma(n)) \), where \( \sigma(n) \) stands for the sum of the divisors function, is 1.

Given an arbitrary integer \( k \geq 0 \), let \( \sigma_k(n) \) stand for the \( k \)-fold iterate of the \( \sigma(n) \) function, that is, \( \sigma_0(n) = n, \sigma_1(n) = \sigma(n), \sigma_2(n) = \sigma(\sigma_1(n)) \), and so on. Here, given any integer \( k \geq 0 \) and any real \( \varepsilon > 0 \), we show that \( \lambda(\sigma_k(n)) < 1 + \varepsilon \) for almost all positive integers \( n \).

We denote by \( p(n) \) and \( P(n) \) the smallest and largest prime factors of \( n \), respectively. We write \( \Pi(n) \) for the largest prime power dividing \( n \). The letters \( p, q, \pi \) and \( Q \), with or without subscript, will stand exclusively for primes. On the other hand, the letters \( c \) and \( C \), with or without subscript, will stand for absolute constants but not necessarily the same at each occurrence. Moreover, we shall use the abbreviations \( x_1 = \log x, x_2 = \log \log x \), and so on. Finally, given any real number \( x \geq 1 \), we let \( \mathcal{N}_x = \{1, 2, \ldots, \lfloor x \rfloor \} \).

2 Main results

**Theorem 2.1.** Given a fixed integer \( k \geq 0 \) and an arbitrary real number \( \varepsilon > 0 \),

\[
\frac{1}{x} \#\{n \leq x : \lambda(\sigma_k(n)) \geq 1 + \varepsilon\} \to 0 \quad (x \to \infty).
\]

**Remark 2.2.** The case \( k = 0 \), namely that the normal order of \( \lambda(n) \) is one, was proved by De Koninck and Doyon [DD]. The case \( k = 1 \) was settled by De Koninck and Luca [DL], who actually proved more, namely that

\[
\frac{1}{x} \sum_{n \leq x} \lambda(\sigma(n)) \to 1 \quad (x \to \infty).
\]

We could not generalize the approach used in [DL] to prove (2.1) for any \( k \geq 2 \). We will therefore use a totally different approach.

On the other hand, letting \( \phi \) stand for the Euler totient function and, given an integer \( k \geq 0 \) and letting \( \phi_k(n) \) stand for the \( k \)-fold iterate of \( \phi(n) \), it turns out that the next theorem is much easier to prove than Theorem 2.1.

**Theorem 2.3.** Given a fixed integer \( k \geq 0 \) and an arbitrary real number \( \varepsilon > 0 \),

\[
\frac{1}{x} \#\{n \leq x : \lambda(\phi_k(n)) \geq 1 + \varepsilon\} \to 0 \quad (x \to \infty).
\]
Finally, let $\sigma^*(n)$ stand for the sum of the unitary divisors of $n$, and for each integer $k \geq 0$, let $\sigma_k^*(n)$ stand for the $k$-fold iterate of the $\sigma^*$ function. We can then prove the following.

**Theorem 2.4.** Given a fixed integer $k \geq 0$ and an arbitrary real number $\varepsilon > 0$,

$$\frac{1}{x} \#\{n \leq x : \lambda(\sigma_k^*(n)) \geq 1 + \varepsilon\} \to 0 \quad (x \to \infty).$$

### 3 Preliminary lemmas

**Lemma 3.1.** For all integers $k$ and $\ell$, let

$$\delta(x, k, \ell) := \sum_{p \leq x \atop p \equiv \ell \pmod{k}} \frac{1}{p}.$$

Then, for $\ell = 1$ or $-1$, $k \leq x$, and $x \geq 3$, we have

$$\delta(x, k, \ell) \leq \frac{C_1 x_2}{\phi(k)},$$

where $C_1 > 0$ is an absolute constant.

**Proof.** This is Lemma 2.5 in Bassily, Kátai and Wijsmuller [BKW].

We say that a $k+1$-tuple of primes $(q_0, q_1, \ldots, q_k)$ is a $k$-chain if $q_{i-1} | q_i + 1$ for $i = 1, 2, \ldots, k$, in which case we write $q_0 \rightarrow q_1 \rightarrow \cdots \rightarrow q_k$. We shall need the following result.

**Lemma 3.2.** For any fixed prime $q_0$ and integer $k \geq 1$, there exist absolute constants $c_1, \ldots, c_k$ such that

$$\sum_{q_1 \leq x \atop q_0 \rightarrow q_1} \frac{1}{q_0} \leq \frac{c_1 x_2}{q_0}, \quad \sum_{q_2 \leq x \atop q_0 \rightarrow q_1 \rightarrow q_2} \frac{1}{q_2} \leq \frac{c_2 x_2^2}{q_0}, \quad \cdots, \quad \sum_{q_k \leq x \atop q_0 \rightarrow q_1 \rightarrow \cdots \rightarrow q_k} \frac{1}{q_k} \leq \frac{c_k x_2^k}{q_0}.$$

**Proof.** Using Lemma 3.1, we have, for some constant $c_1 > 0$,

$$\sum_{q_1 \leq x \atop q_0 \rightarrow q_1} \frac{1}{q_1} \leq \frac{C_1 x_2}{\phi(q_0)} = \frac{C_1 x_2}{q_0 - 1} \leq \frac{c_1 x_2}{q_0},$$

which proves the first inequality. To obtain the second one, observe that, using (3.1), for some constant $c_2 > 0$,

$$\sum_{q_2 \leq x \atop q_0 \rightarrow q_1 \rightarrow q_2} \frac{1}{q_2} = \sum_{q_1 \leq x \atop q_0 \rightarrow q_1} \sum_{q_2 \leq x \atop q_0 \rightarrow q_1 \rightarrow q_2} \frac{1}{q_2} \leq \sum_{q_1 \leq x \atop q_0 \rightarrow q_1} \frac{c_1 x_2}{q_1} = c_1 x_2 \sum_{q_1 \leq x \atop q_0 \rightarrow q_1} \frac{1}{q_1} \leq c_1 x_2 \frac{c_1 x_2}{q_0} = \frac{c_2 x_2^2}{q_0},$$

thus establishing the second inequality. Proceeding in the same manner, the proof of the other inequalities is straightforward. \qed
4 Proof of the theorems

We only prove Theorem 2.1 since the proofs of Theorems 2.3 and 2.4 are similar.

We first introduce the sequence \((w_k)_{k \geq 0} = (w_k(x))_{k \geq 0}\) defined as the real function satisfying

\[
\text{(4.1)} \quad \log w_k(x) = x^{m_k},
\]

where \(0 < m_0 < m_1 < \cdots\) is a suitable sequence of integers, which is to be determined later.

Our plan is to introduce our approach in the cases \(k = 0\) and \(k = 1\) and then to use induction on \(k\).

We first examine the cases \(k = 0\) and \(k = 1\). In the case \(k = 0\), we first write each positive integer \(n \leq x\) as

\[
\sigma_0(n) = n = A_0(n)B_0(n),
\]

where \(B_0(n) := \prod_{q | n} q\) and \(A_0(n) = n/B_0(n)\). Then, let \(Y_x \to \infty\) as \(x \to \infty\) with \(Y_x \leq x_5\) and consider the set

\[
\mathcal{U}_x^{(0)} := \{n \in \mathbb{N} : \mu(B_0(n)) = 0 \text{ or } \Pi(A_0(n)) > Y_x^{Y_x} \text{ or } P(A_0(n)) \geq Y_x\},
\]

where \(\mu\) stands for the Moebius function, observing that

\[
\text{(4.2)} \quad \#\mathcal{U}_x^{(0)} = o(x) \quad (x \to \infty).
\]

Now setting

\[
\mathcal{N}_x^{(1)} := \mathcal{N} \setminus \mathcal{U}_x^{(0)},
\]

we have that, for \(n \in \mathcal{N}_x^{(1)}\), \(B_0(n)\) is squarefree and \((A_0(n), B_0(n)) = 1\), thus allowing us to write

\[
\sigma(n) = \sigma(A_0(n))\sigma(B_0(n)) \quad (n \in \mathcal{N}_x^{(1)}).
\]

To each prime number \(q\), let us associate the strongly additive function \(f_q\) defined on primes \(p\) by

\[
f_q(p) = \begin{cases} k & \text{if } q^k \Vert p + 1, \\ 0 & \text{if } q \nmid p + 1. \end{cases}
\]

Then, we set

\[
s(n) := \prod_{q \leq x_2^2} q^{f_q(n)}
\]
and
\[ E(x) := \sum_{n \in \mathcal{N}_x^{(1)}} \log s(n) = \sum_{n \in \mathcal{N}_x^{(1)}} \sum_{q \leq x^2} (\log q) f_q(n). \]

We have, in light of Lemma 3.1,
\[ E(x) \leq \sum_{q \leq x^2} (\log q) \sum_{q^k \leq x} \frac{1}{p} \leq C_1 xx_2 \sum_{q \leq x^2} \frac{\log q}{\phi(q^k)} \leq C_2 x x_2 x_3, \]
so that
\[ (4.3) \quad s(n) < \exp(x_2 x_3 x_4) \quad \text{for } n \leq x \text{ with at most } o(x) \text{ exceptions.} \]

Letting \( \mathcal{U}_x^{(1)} \) be the set of those integers \( n \in \mathcal{N}_x^{(1)} \) for which \( q^2 \mid \sigma(B_0(n)) \) for at least one prime \( q > x^2 \), we have, using Lemma 3.2,
\[ \sum_{n \in \mathcal{N}_x^{(1)}} \sum_{q \leq w_1^{(1)}} \frac{1}{q^2} \leq \sum_{x^2 < q} \frac{x}{\phi(q^2)} + \sum_{x^2 < q} \left( \sum_{p \mid q^2} \frac{x}{p} \right) \]
\[ \leq C_3 x x_2 \sum_{q > x^2} \frac{1}{q^2} \leq C_4 x \frac{x}{x_3}, \]
implying that
\[ (4.4) \quad \# \mathcal{U}_x^{(1)} = o(x) \quad (x \to \infty). \]

Letting \( w_1 = w_1(x) \) be such that \( \log w_1(x) = x^2 \) (that is, choosing \( m_1 = 2 \) in (4.1)) and setting
\[ r(n) := \prod_{q \mid \sigma(B_0(n)) \atop x^2 < q \leq w_1} q, \]
we have, again using Lemma 3.2, that
\[ \sum_{n \leq x} \log r(n) \leq \sum_{x^2 < q \leq w_1} (\log q) \sum_{p \leq x} \frac{x}{q \cdot p} \]
\[ \leq C_5 x x_2 \sum_{q \leq w_1} \frac{\log q}{q} \leq C_6 x x_2^3. \]

We now set \( \mathcal{N}_x^{(2)} := \mathcal{N}_x^{(1)} \setminus \mathcal{U}_x^{(1)} \) and
\[ \mathcal{U}_x^{(2)} := \mathcal{N}_x^{(2)} : s(n) > \exp(x_2 x_3 x_4) \text{ or } r(n) > \exp(x_2^3 x_3) \} \).

Thus, in light of (4.2), (4.4), (4.3) and (4.5), we have that
\[ \# \mathcal{U}_x^{(2)} = o(x) \quad (x \to \infty). \]
This motivates the definition

\[ N^{(3)}_x := N^{(2)}_x \setminus U^{(2)}_x. \]

Writing

\[
A_1(n) = \sigma(A_0(n)) \cdot s(n) \cdot r(n), \\
B_1(n) = \prod_{q \mid \sigma(B_0(n))} q, \\
\]

we then have that

\[ A_1(n) \leq \sigma(A_0(n)) \exp(2x_2^3x_3) \quad (n \in N^{(3)}_x). \tag{4.6} \]

On the other hand,

\[ \sigma(A_0(n)) \leq CY_x^Y \log Y_x^{Y_x} \leq x_3 \quad (n \in N^{(3)}_x), \tag{4.7} \]

which implies that \((\sigma(A_0(n)), B_1(n)) = 1\), and since we obviously have \((s(n)r(n), B_1(n)) = 1\), we may conclude that

\[ \sigma(n) = A_1(n)B_1(n), \]

where

\[ (A_1(n), B_1(n)) = 1, \quad B_1(n) \text{ is squarefree}, \quad B_1(n) \mid \gamma(\sigma(n)). \]

Consequently, in light of (4.6) and (4.7), we have

\[ \frac{\sigma(n)}{\gamma(\sigma(n))} \leq A_1(n) \leq x_3 \exp(2x_2^3x_3) \quad (n \in N^{(3)}_x). \tag{4.8} \]

Now, write

\[ \lambda(\sigma(n)) = \frac{\log \sigma(n)}{\log \gamma(\sigma(n))} = 1 + \frac{\log(\sigma(n)/\gamma(\sigma(n)))}{\log \gamma(\sigma(n))} = 1 + \theta_n, \tag{4.9} \]

say. Using (4.6) and (4.8), it follows that

\[ \theta_n \leq \frac{\log(\sigma(n)/\gamma(\sigma(n)))}{\log(\sigma(n)/A_1(n))} \leq \frac{x_4 + 2x_2^3x_3}{\log \sigma(n) - (x_4 + 2x_2^3x_3)}. \tag{4.10} \]

Since, for \( n \in [x/x_1, x] \), we have that \( \log \sigma(n) > x_1 - x_2 \), it follows from (4.10) that

\[ \theta_n \leq \frac{3x_2^2x_3}{x_1} \quad (n \in N^{(3)}_x). \tag{4.11} \]
Using (4.11) in (4.9) proves (2.1) for the case \( k = 1 \).

Having proved our result for the cases \( k = 0 \) and \( k = 1 \), we now use induction. Indeed, assuming that (2.1) is true for \( j = 1, 2, \ldots, k \), we will prove that it holds for \( j = k + 1 \). Then, for \( j = 1, \ldots, k \), we let \( \mathcal{N}_x^{(j)} \) be the sets with \( \mathcal{N}_x \supseteq \mathcal{N}_x^{(1)} \supseteq \mathcal{N}_x^{(2)} \supseteq \cdots \) and

\[
\sigma_j(n) = A_j(n) \cdot B_j(n) \quad (n \in \mathcal{N}_x^{(j)}),
\]

where

\[
B_j(n) = \prod_{q \mid \sigma(B_{j-1}(n))} q \quad \text{and} \quad A_j(n) = \frac{\sigma_j(n)}{B_j(n)},
\]

with \( w_j = w_j(x) \) as in (4.1) and

\[
\sigma(A_j(n)) < w_{j+1} \quad (n \in \mathcal{N}_x^{(j)}),
\]

with

\[
(A_j(n), B_j(n)) = 1, \quad B_j(n) \text{ is squarefree,} \quad p(B_j(n)) > w_j.
\]

We can therefore write

\[
\sigma_k(n) = A_k(n)B_k(n),
\]

where

\[
\sigma(A_k(n)) < w_{k+1}, \quad p(B_k(n)) > w_k,
\]

\( B_k(n) \) is squarefree and \( (A_k(n), B_k(n)) = 1 \).

Hence, we have that \( B_k(n) \) is a divisor of \( \gamma(\sigma_k(n)) \) and therefore, following the same argument as in the case \( k = 1 \), we obtain that (2.1) holds for \( k \).

For the case \( k + 1 \), we first write

\[
(4.12) \quad \sigma_{k+1}(n) = \sigma(A_k(n))\sigma(B_k(n))
\]

and set

\[
s_k(n) = \prod_{\pi \mid \sigma(B_k(n))} \pi f_\pi(\sigma(B_k(n))),
\]

\[
r_k(n) = \prod_{\pi \mid \sigma(B_k(n))} \pi,
\]

\[
t_k(n) = \prod_{\pi \mid \sigma(B_k(n))} \pi,
\]

where \( \pi \) ranges over primes not exceeding \( w_{k+1} \).
where, in each of the above products, \( \pi \) runs over primes. First observe that
\[
\sum_{n \in \mathcal{N}_x^{(k)}} \log s_k(n) = \sum_{n \in \mathcal{N}_x^{(k)}} \sum_{\pi \leq x_2^{2(k+1)}} f_\pi(\sigma(B_k(n))) \log \pi \\
= \sum_{\pi \leq x_2^{2(k+1)}} (\log \pi) \sum_{\pi \rightarrow p_1 \rightarrow \cdots \rightarrow p_k+1, p_1 > x_k+1} f_\pi(p_1) \left\lfloor \frac{x}{p_k+1} \right\rfloor \\
\leq C_{k+1} x x_2^{2(k+1)} \sum_{\pi \leq x_2^{2(k+1)}} \frac{\log \pi}{\pi} \\
\leq C_{k+1} x x_2^{2(k+1)} x_3.
\]

It follows from this estimate that
\[
\frac{1}{x} \# \{ n \in \mathcal{N}_x^{(k)} : s_k(n) > e^{\kappa_x x_2^{2(k+1)} x_3} \} \rightarrow 0 \quad (x \rightarrow \infty),
\]
provided \( \kappa_x \) is any function such that \( \kappa_x \rightarrow \infty \) arbitrarily slowly as \( x \rightarrow \infty \).

We will now prove that, as \( x \rightarrow \infty \),
\[
\frac{1}{x} \# \{ n \in \mathcal{N}_x^{(k)} : \text{there exists } \pi > x_2^{2(k+1)} \text{ such that } \pi^2 | \sigma(B_k(n)) \} \rightarrow 0.
\]

Indeed, if \( n \in \mathcal{N}_x^{(k)} \) and \( \pi^2 | \sigma(B_k(n)) \), we then have that there exist two chains of primes, namely
\[
\pi \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{k+1}, \quad Q_{k+1} | n, \\
\pi \rightarrow p_1 \rightarrow \cdots \rightarrow p_{k+1}, \quad p_{k+1} | n,
\]
from which it follows, using Lemma 3.2, that
\[
\sum_{\pi > x_2^{2(k+1)}} \sum_{n \in \mathcal{N}_x^{(k)}} \frac{1}{\pi^2 | \sigma(B_k(n))} \leq C_{k+1} x x_2^{2(k+1)} \sum_{\pi > x_2^{2(k+1)}} \frac{1}{\pi^2} \ll \frac{x}{x_3},
\]
thus establishing (4.14).

On the other hand,
\[
\sum_{n \in \mathcal{N}_x^{(k)}} \log r_k(n) = \sum_{x_2^{2(k+1)} < \pi < w_{k+1}} (\log \pi) \sum_{\pi \rightarrow p_1 \rightarrow \cdots \rightarrow p_k+1} \left\lfloor \frac{x}{p_k+1} \right\rfloor \\
\leq C_{k+1} x x_2^{2(k+1)} \sum_{\pi < w_{k+1}} \frac{\log \pi}{\pi} \leq C_{k+1} x x_2^{2(k+1)} \log w_{k+1} \\
= C_{k+1} x x_2^{2(k+1)+m_{k+1}}.
\]
which proves that

\[(4.15) \quad \frac{1}{x} \# \left\{ n \in \mathcal{N}_x^{(k)} : r_k(n) > e^{C_x^{2(k+1)+m_{k+1}}} \right\} \to 0 \quad (x \to \infty).\]

Replacing (4.12) by

\[\sigma_{k+1}(n) = \sigma(A_k(n))s_k(n)r_k(n)t_k(n),\]

then, since for all \(n \in \mathcal{N}_x^{(k)},\) we have \(\sigma(A_k(n)) < w_{k+1}\) while \(p(t_k(n)) > w_{k+1}\) and \(P(s_k(n)r_k(n)) < w_{k+1},\) it follows that, choosing

\[A_{k+1}(n) = \sigma(A_k(n))s_k(n)r_k(n),\]
\[B_{k+1}(n) = t_k(n),\]

we have

\[\sigma_{k+1}(n) = A_{k+1}(n)B_{k+1}(n).\]

In light of (4.13), (4.14) and (4.15), we can now say that, with the possible exception of \(o(x)\) integers \(n \leq x\) as \(x \to \infty,\)

\[\sigma(A_{k+1}(n)) < w_{k+2}\]

for a corresponding suitable large integer \(m_{k+2}.\)

Moreover, since \(B_{k+1}(n)\) is squarefree, we obtain that

\[B_{k+1}(n) \mid \gamma(\sigma_{k+1}(n)),\]

and we may then conclude the proof similarly as in the case of \(k,\) thus proving (2.1) for the case \(k+1\) and thereby completing the proof of Theorem 2.1.

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**References**


