ARITHMETIC FUNCTIONS AND THEIR COPRIMALITY
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Abstract: Let $D \geq 3$ be an odd integer and $\ell \geq -1$ be a non-zero integer such that $\gcd(\ell, D) = 1$. Let $f, g : \mathbb{N} \to \mathbb{N}$ be multiplicative functions such that $f(p) = D$ and $g(p) = p + \ell$ for each prime $p$. We estimate the number of positive integers $n \leq x$ such that $\gcd(f(n), g(n)) = 1$. If $D$ is a prime larger than 3, we also examine the size of the number of positive integers $n \leq x$ for which $\gcd(g(n), f(n - 1)) = 1$.

Keywords: Arithmetic functions, number of divisors, sum of divisors, shifted primes.

1. Introduction

Given an arithmetical function $f$ and a large number $x$, examining the number of positive integers $n \leq x$ for which $\gcd(n, f(n)) = 1$, has been the focus of several papers. For instance, Paul Erdős [4] established that

$$
\#\{n \leq x : \gcd(n, \varphi(n)) = 1\} = (1 + o(1)) \frac{e^{-\gamma}x}{\log\log\log x} \quad (x \to \infty),
$$

where $\varphi$ is the Euler function and $\gamma$ is the Euler constant. A similar result can be obtained if one replaces $\varphi(n)$ by $\sigma(n)$, the sum of the divisors of $n$. Similarly, letting $\Omega(n)$ stand for the number of prime factors of $n$ counting their multiplicity, Alladi [1] proved that the probability that $n$ and $\Omega(n)$ are relatively prime is equal to $6/\pi^2$ by examining the size of $\{n \leq x : \gcd(n, \Omega(n)) = 1\}$. Let $K(x)$ stand for the number of positive integers $n \leq x$ such that $\gcd(n\tau(n), \sigma(n)) = 1$, where $\tau(n)$ stands for the number of divisors of $n$. Some fifty years ago, Kanold [5] showed that there exist positive constants $c_1 < c_2$ and a positive number $x_0$ such that

$$
c_1 < K(x)/\sqrt{x/\log x} < c_2 \quad (x \geq x_0).
$$

In 2007, the authors [2] proved that there exists a positive constant $c_3$ such that

$$
K(x) = c_3(1 + o(1)) \sqrt{\frac{x}{\log x}} \quad (x \to \infty).
$$

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The analogue problem for counting the number of positive integers \( n \) for which

\[
gcd(n\tau(n), \varphi(n)) = 1 \tag{1.1}
\]
is trivial. Clearly (1.1) holds for \( n = 1, 2 \). But these are the only solutions. Indeed, assume that (1.1) holds for some \( n \geq 3 \). Then \( n \) is squarefree and it must therefore have an odd prime divisor \( p \), in which case \( 2|\varphi(n) \) and \( 2|\tau(n) \), implying that \( gcd(n\tau(n), \varphi(n)) > 1 \), thereby proving our claim.

More recently, we obtained (see [3]) asymptotic estimates for the counting functions

\[
R(x) := \#\{n \leq x : gcd(\varphi(n), \tau(n)) = gcd(\sigma(n), \tau(n)) = 1\}
\]
and

\[
N(x) := \#\{n \leq x : \ell(n) = 1\},
\]
where \( \ell(n) := gcd(\tau(n), \tau(n + 1)) \). In fact, we proved that, as \( x \to \infty \),

\[
R(x) = (c_4 + o(1))\sqrt{\frac{x}{\log x}} \quad \text{and} \quad N(x) = (c_5 + o(1))\sqrt{x},
\]
where \( c_4 \) and \( c_5 \) are positive constants.

Let \( D \geq 3 \) be an odd integer and let \( \ell \geq -1 \) be a non zero integer such that \( gcd(\ell, D) = 1 \). Let \( f, g : \mathbb{N} \to \mathbb{N} \) be multiplicative functions such that \( f(p) = D \) and \( g(p) = p + \ell \) for each prime \( p \). In this paper, we estimate the number \( E(x) \) of positive integers \( n \leq x \) such that

\[
gcd(f(n), g(n)) = 1. \tag{1.2}
\]

Our general result will apply in particular to the case \( g(n) = \varphi(n) \) (or \( \sigma(n) \)) and \( f(n) = \tau_k(n) \) with \( k \) odd, \( k \geq 3 \), where \( \tau_k(n) \) stands for the number of ways one can write \( n \) as the product of \( k \) positive integers taking into account the order in which they are written. Another valid choice is \( f(n) = k^{\omega(n)} \) with \( k \) odd, \( k \geq 3 \), where \( \omega(n) \) stands for the number of distinct prime factors of \( n \) with \( \omega(1) = 0 \).

Moreover, in the case where \( D > 3 \) is a prime, we shall also examine the size of the number \( S(x) \) of positive integers \( n \leq x \) for which

\[
Z(n) := gcd(g(n), f(n - 1)) = 1.
\]

From here on, \( gcd(a, b) \) will be written simply as \((a, b)\). In what follows, we shall denote the logarithmic integral of \( x \) by \( li(x) \), that is \( li(x) := \int_{2}^{x} \frac{dt}{\log t} \), while \( \Gamma \) stands for the Gamma function. We say that a positive integer \( n \) is squarefull if \( p^2|n \) for all prime divisors \( p \) of \( n \); we will denote by \( \mathcal{F} \) the set of squarefull numbers. Moreover, the letters \( c \) and \( C \) will stand for positive constants, while the letters \( p \) and \( q \) will always stand for prime numbers. Finally, given any set of positive integers \( \mathcal{B} \), the expression \( N(\mathcal{B}) \) stands for the multiplicative semi-group generated by \( \mathcal{B} \).

Finally, given \( D \) and \( \ell \) as above, we let \( t_1, t_2, \ldots, t_T \) be all those reduced residue classes mod \( D \) for which \( (t_j + \ell, D) = 1 \) for \( j = 1, 2, \ldots, T \).
2. Main results

**Theorem 2.1.** There exists a positive constant $c_6$ such that
\[
E(x) = (c_6 + o(1))x \log^\tau x \quad (x \to \infty),
\]
where $\tau = T/\varphi(D)$.

**Theorem 2.2.** There exists a positive constant $c_7$ such that
\[
S(x) = (c_7 + o(1))x \log^\tau x \quad (x \to \infty),
\]
where, in this case, $\tau - 1 = -1/(D - 1)$.

3. Preliminary results

To prove our results we shall need the following results.

**Theorem A (Wirsing).** Let $f$ be a non negative multiplicative function for which there exist two positive constants $a_1$ and $a_2 < 2$ such that $f(p^\alpha) \leq a_1 a_2^\alpha$ for each integer $\alpha \geq 2$. Assume also that there exists a positive constant $C$ such that
\[
\sum_{p \leq x} f(p) = (C + o(1)) \frac{x}{\log x} \quad (x \to \infty).
\]
Then
\[
\sum_{n \leq x} f(n) = \left( \frac{e^{-\gamma C}}{\Gamma(C)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots \right) \quad (x \to \infty).
\]

**Theorem B (Levin and Feinleib).** Let $f$ be a complex valued multiplicative function satisfying the three conditions
\[
\sum_{p \leq x} f(p) = (C + o(1)) \frac{x}{\log x} \quad (x \to \infty),
\]
\[
\sum_{p \leq x} |f(p)| = O\left( \frac{x}{\log x} \right),
\]
\[
f(p^r) = O((2p)^{c_0 r}),
\]
where $C$ and $c_0$ are positive constants with the additional restriction $c_0 < 1/2$.
Then,
\[
\sum_{n \leq x} f(n) = e^{-\gamma C} \frac{x}{\Gamma(C) \log x} \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots \right) + o \left( \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{|f(p)|}{p} + \frac{|f(p^2)|}{p^2} + \ldots \right) \right) \quad (x \to \infty).
\]

**Proofs.** The results of Theorems A and B can be found in Chapter 4 of the book of Postnikov [6].
4. The proof of Theorem 2.1

Let $\wp_{\ell, D}$ be the set of primes $p$ for which $p \equiv t_j \pmod{D}$ for $j = 1, \ldots, T$. Furthermore, let $H = \{p : p|D\}$ and set

$\wp_{\ell, D, H} = \wp_{\ell, D} \cup H$.

It is well known that

$$\# \{n \leq x : n \in \mathcal{F}\} = O(\sqrt{x}) \quad (x \to \infty). \quad (4.1)$$

Hence, given a non squarefull integer $n \leq x$, let us write it as $n = Km$, where $K \in \mathcal{F}$ and $m > 1$ is squarefree with $(K, m) = 1$, so that condition (1.2) can be written as

$$(f(K)f(m), g(K)g(m)) = 1. \quad (4.2)$$

So, for each $K \in \mathcal{F}$, let us set

$$E_K(x) := \# \{n = Km \leq x : m > 1 \text{ and } (4.2) \text{ holds}\},$$

so that, in light of (4.1),

$$E(x) = \sum_{K \in \mathcal{F}} E_K(x) + O(\sqrt{x}). \quad (4.3)$$

By using the Brun-Selberg Sieve, we obtain that for $1 \leq K \leq \sqrt{x}$,

$$E_K(x) \leq E_1 \left(\frac{x}{K}\right) \ll \frac{x}{K} \prod_{\substack{p \leq \sqrt{x} \atop p \not\in \wp_{\ell, D}}} \left(1 - \frac{1}{p}\right) \ll \frac{x}{K} (\log x)^{\tau - 1}, \quad (4.4)$$

while we trivially have that $E_K(x) \ll x/K$ if $\sqrt{x} < K \leq x$. Since $\sum_{K \in \mathcal{F}} \frac{1}{K}$ is convergent, it follows from (4.3) and (4.4) that

$$E(x) = \sum_{K \in \mathcal{F} \atop K < Y_x} E_K(x) + o(x(\log x)^{\tau - 1}) \quad (x \to \infty), \quad (4.5)$$

where $Y_x$ is an arbitrary function tending to infinity as $x \to \infty$, which we can also assume to satisfy $\max_{n \leq Y_x} f(n) \leq \log \log \log x$, say.

Observe that a necessary condition for (4.2) to hold is that

$$(g(K), f(K)D) = 1. \quad (4.6)$$

Now let $\mathcal{K}$ be the set of those $K \in \mathcal{F}$ for which (4.6) holds. Note that the set $\mathcal{K}$ is non empty, since $1 \in \mathcal{K}$. Moreover, define

$$\mathcal{K}_0 = \{K \in \mathcal{K} : p|f(K) \Rightarrow p|D\}.$$
We shall prove that, for every fixed $K \in \mathcal{F}$, as $x \to \infty$,

$$E_K(x) = o(E_1(x)) \quad \text{if } K \in \mathcal{F} \setminus \mathcal{K}_0,$$

$$E_K(x) = (1 + o(1)) \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} E_1(x) \quad \text{if } K \in \mathcal{K}_0,$$

$$E_1(x) = (c + o(1))x \log \tau - 1 x,$$

(4.7) (4.8) (4.9)

where $c$ is a positive constant. Combining these three estimates with (4.4) and (4.5), Theorem 2.1 will follow immediately.

For a given $K \in \mathcal{K}_0$, letting $E(y, K) := \#\{m \in [2, y] : m \text{ squarefree and } (m, K) = 1\}$, it is clear that the number of positive integers $n = K m \leq x$, with $m > 1$, for which (4.2) holds is equal to $E(x/K, K)$.

Consider the multiplicative function $h_K$ defined on prime powers by $h_K(p^\alpha) = 0$ if $\alpha \geq 2$ or if $p \mid K$, and by

$$h_K(p) = \begin{cases} 1 & \text{if } p \nmid K \text{ and } p \in \mathcal{P}_{\ell,D,H}, \\ 0 & \text{otherwise}. \end{cases}$$

With this definition of $h_K$, we have that

$$E(y, K) = \sum_{n \leq y} h_K(n).$$

(4.10)

To estimate this last sum, we shall consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{h_K(n)}{n^s} = \prod_p \left(1 + \frac{h_K(p)}{p} + \frac{h_K(p^2)}{p^2} + \ldots\right)$$

$$= \prod_{p \mid K} \left(1 + \frac{1}{p^s}\right)^{-1} \prod_{p \in \mathcal{P}_{\ell,D,H}} \left(1 + \frac{1}{p^s}\right).$$

In light of the fact that

$$\sum_{p \leq x} h_K(p) = (\tau + o(1)) \frac{x}{\log x} \quad (x \to \infty),$$

we may use Theorem A and obtain that, as $x \to \infty$,

$$E_K(x) = (1 + o(1)) \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} \frac{x}{\log x} \exp\{\tau \log \log x + C_D + o(1)\},$$

$$\left[(1 + o(1)) \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} \frac{x}{\log x} \exp\{\tau \log \log x + C_D + o(1)\} = \right]$$

$$\left[\prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} \frac{x}{\log x} \exp\{\tau \log \log x + C_D + o(1)\}\right].$$
where $C_D$ is a suitable constant depending only on $D$. Estimates (4.8) and (4.9) are thus established. It remains to prove (4.7). So, let $K \in \mathcal{F} \setminus \mathcal{K}_0$ be fixed. Then, there exists a prime divisor $q$ of $f(K)$ such that $(q, D) = 1$. If (4.2) holds, then the fact that $p|m$ implies that $(p + \ell, qD) = 1$. Hence, from the Brun-Selberg Sieve, it follows that

$$E_K(x) \ll \frac{x}{K} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p} - \sum_{\substack{p \leq x \atop (p + \ell, qD) > 1}} \frac{1}{p}\right) \left(1 + \frac{1}{p} - \sum_{\substack{p \leq x \atop (p + \ell, D) = 1 \atop q | p + \ell}} \frac{1}{p}\right)$$

$$\ll \frac{x}{K} \exp \left\{ - \sum_{p \leq x} \frac{1}{p} - \sum_{\substack{p \leq x \atop (p + \ell, D) = 1 \atop q | p + \ell}} \frac{1}{p}\right\}$$

$$\ll \frac{x}{K} \exp \left\{ - \left(1 - \frac{T}{\varphi(D)}\right) \log \log x - \frac{T}{\varphi(D)q - 1} \log \log x\right\}$$

$$\ll \frac{x}{K} \log^{\tau-1} x \cdot \exp \left(- \frac{T}{\varphi(D)(q - 1)} \log \log x\right),$$

thereby implying that (4.7) holds and thus completing the proof of Theorem 2.1.

5. The proof of Theorem 2.2

First observe that the number of those integers $n \leq x$ for which $n$ or $n - 1$ is a squarefull number is $O(\sqrt{x})$.

Let us write $n = Km$ and $n - 1 = R\nu$, where $K$ and $R$ are squarefull, while $m$ and $\nu$ are squarefree, with $(K, m) = 1$ and $(R, \nu) = 1$. Then, for each pair of coprime squarefull numbers $K$ and $R$, define

$$S_{K, R}(x) = \#\{n \leq x : n = Km, \ n - 1 = R\nu, \ m > 1, \ \nu > 1, \ Z(n) = 1\}.$$

With these notations and the above observation, it is clear that

$$S(x) = \sum_{K, R \in \mathcal{F}} S_{K, R}(x) + O(\sqrt{x}). \quad (5.1)$$

Since in this case, $H = D$, it follows that if $n = Km$, $n - 1 = R\nu$, $m > 1$, $\nu > 1$ and $Z(n) = 1$, then $\nu \in \mathcal{P}_{\ell, D, D}$. Consequently, by using the Brun-Selberg Sieve, we obtain that, for each squarefull number $R$,

$$\sum_{K \in \mathcal{F}} S_{K, R}(x) \ll \begin{cases} \frac{x}{R} \log^{\tau-1} x & \text{if } R \leq \sqrt{x}, \\ \frac{x}{R} & \text{if } \sqrt{x} < R \leq x. \end{cases} \quad (5.2)$$

Fixing $K \in \mathcal{F}$, we shall estimate the number of integers $n \leq x$ such that $K|n$ and for which $n - 1 = R\nu$ with $R \in \mathcal{F}$ and $\nu \in \mathcal{N}(\mathcal{P}_{\ell, D, D})$. 


Similarly as in (5.2), we have
\[
\sum_{R \in \mathcal{F}} S_{K,R}(x) \ll \begin{cases} 
\frac{x}{K} \log^{\tau-1} x & \text{if } K \leq \sqrt{x}, \\
\frac{x}{K} & \text{if } \sqrt{x} < K \leq x.
\end{cases} \tag{5.3}
\]

It follows from (5.2) and (5.3) that for an arbitrary function \( Y_x \to \infty \),
\[
\sum_{\max(K,R) > Y_x} S_{K,R}(x) = o\left(x \log^{\tau-1} x\right) \quad (x \to \infty). \tag{5.4}
\]

So, let us assume that \( \max(K, R) \leq Y_x \) and define
\[
\mathcal{R}_0 = \{ R \in \mathcal{F} : q | f(R) \Rightarrow q = D \} \quad \text{and} \quad \mathcal{R}_1 = \mathcal{F} \setminus \mathcal{R}_0.
\]

Fix \( R \in \mathcal{R}_1 \) and let \( q | f(R) \) with \( q \neq D \). Then, \( n = K \ell m \) implies that \( R \ell + 1 \equiv 0 \pmod{K} \), while \( \ell(n) = 1 \) implies that \( (g(m), Dq) = 1 \). Thus, by using the Brun-Selberg Sieve, we have that
\[
\sum_{K \in \mathcal{F}} S_{K,R}(x) = o\left(\frac{x}{R} \log^{\tau-1} x\right) \quad (x \to \infty),
\]
so that
\[
\sum_{R \in \mathcal{R}_1} \sum_{K \in \mathcal{F}} S_{K,R}(x) = o\left(x \log^{\tau-1} x\right) \quad (x \to \infty). \tag{5.5}
\]

We will say that \( K, R \) is an admissible pair if \((g(K), D f(R)) = 1\). Observe that it is clear that \( S_{K,R}(x) = 0 \) if \( K, R \) is not an admissible pair, and also that in the case \( R \in \mathcal{R}_0 \), \((g(K), D f(R)) = 1\) is equivalent to \((g(K), D) = 1\). Finally, observe that \( K = 1, R = 1 \) is an admissible pair.

From (5.4), (5.5) and (5.1), it therefore follows that
\[
S(x) = \sum_{\substack{R \in \mathcal{R}_0 \\text{admissible pair} \\text{max}(K,R) \leq Y_x}} S_{K,R}(x) + o(x \log^{\tau-1} x). \tag{5.6}
\]

Let \( F \) be the multiplicative function defined by
\[
F(p) = \begin{cases} 
1 & \text{if } p + \ell \not\equiv 0 \pmod{D}, \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
F(p^\alpha) = 0 \text{ if } \alpha \geq 2,
\]
and define the function \( m(x) = \prod_{p \leq x} \left(1 + \frac{F(p)}{p}\right) \).

It is clear that if \( 0 < \varepsilon_x \to 0 \text{ as } x \to \infty \), then \( \max_{x^{1-\varepsilon_x} \leq y \leq x} \left| \frac{m(y)}{m(x)} - 1 \right| \to 0 \text{ as } x \to \infty. \)
Given an integer \( B \geq 2 \), let \( \chi_B \) be a character mod \( B \) and assume that \( \chi_B(n_j) = 1 \) for the \( H \) distinct residue classes \( n_j \pmod{B} \). It is clear that if \( H > \varphi(B)/2 \), then \( \chi_B = \chi_B^{(0)} \) is the principal character mod \( B \).

We now define the functions \( u \) and \( V \) as follows.

Set \( u(n) = \chi_K^{(0)}(n)F(n) \), \( n = Km, n - 1 = R\nu \), so that \( u(m) = 1 \) if and only if \((m, K) = 1, m \) is squarefree and \( p|m \) implies that \( p + \ell \not\equiv 0 \pmod{D} \). Let \( V \) be the multiplicative function defined by

\[
V(p) = \begin{cases} 
0 & \text{if } (p, R) = 1, \\
1 & \text{if } p|R, 
\end{cases}
\]

and \( V(p^\alpha) = 0 \) if \( \alpha \geq 3 \).

Observe that if \( V(\delta) \neq 0 \), then we may write \( \delta = \delta_1\delta_2^2 \) with \( \delta_1|R \) and \( (\delta_2, R) = 1 \), so that \( V(\delta) = \mu(\delta_1)\mu(\delta_2) \), where \( \mu \) stands for the Moebius function. Therefore,

\[
V(\delta) = \begin{cases} 
\mu(\delta_1)\mu(\delta_2) & \text{if } (\delta_1, R) = 1 \text{ and } \delta_2|R, \\
0 & \text{otherwise.}
\end{cases}
\]

It follows from this definition that

\[
\sum_{\delta | \nu} V(\delta) = \begin{cases} 
1 & \text{if } (\nu, R) = 1, \nu \text{ squarefree,} \\
0 & \text{otherwise.}
\end{cases}
\]

Now, let \( m_0, \nu_0 \) be the smallest non negative squarefree integers such that

\[
Km_0 - R\nu_0 = 1,
\]

so that all integer solutions of \( Km - R\nu = 1 \) are given by \( m = m_0 + tR \) and \( \nu = \nu_0 + tK \) for \( t \in \mathbb{Z} \).

With the above definitions, we have

\[
S_{K,R}(x) = \sum_{\delta \leq x/R} V(\delta) \sum_{t \leq (x-m_0)/KR} u(m_0 + tR).
\]

If \( (\delta, K) > 1 \), then \((\delta, K) = (\delta_1^2, K) \) and \( \delta_1^2 \) and \( K \) are both squarefull. It follows that if \( p|(\delta_1^2, K) \), then \( p^2|\delta_1^2 \) and \( p^2|K \), so that \( p^2|\nu_0 \) and consequently \( p^2|\nu_0 + tK \) for each \( t \in \mathbb{Z} \), implying that there each number \( \nu_0 + tK \) is squarefull. Hence it follows that in this case, \( S_{K,R}(x) = 0 \). Therefore, we can from now on assume that \( (\delta, K) = 1 \) (which holds if and only if \( (\delta_1, K) = 1 \)).

Since \( (\delta, K) = 1 \), it follows that the congruence \( \nu_0 + tK \equiv 0 \pmod{\delta} \) has one solution mod \( \delta \), represented by \( \nu_0 + t_0K \equiv 0 \pmod{\delta} \), say. This implies that all solutions of the congruence \( \nu_0 + tK \equiv 0 \pmod{\delta} \) are given by \( t = t_0 + k\delta, \, k \in \mathbb{Z} \).
In light of these observations, (5.8) can be written as

\[ S_{K,R}(x) = \sum_{\delta \leq x/R} V(\delta) \sum_{\substack{k \leq (x-(m_0+t_0R))/\delta R}} u(m_0 + t_0R + \delta Rk) \]

\[ = \sum_{\delta \leq x/R} V(\delta) M(\delta) \]

\[ = \sum_{\delta \leq U_x} V(\delta) M(\delta) + \sum_{\delta > U_x} V(\delta) M(\delta) = \Sigma_1 + \Sigma_2, \]

(5.9)
say, where \( U(x) \) is a function chosen so that \( U(x) \to \infty \) as \( x \to \infty \) and \( U(x) = O(\log \log \log x) \).

By the Brun-Selberg Sieve, we obtain that

\[ \Sigma_2 \leq \sum_{(\delta, K) = 1} \sum_{\delta \leq \sqrt{x}} |V(\delta)| \frac{x}{\delta R} \prod_{p \leq x \atop \delta \equiv \ell \pmod{D}} \left(1 - \frac{1}{p}\right) + \sum_{\delta > U_x} \frac{x}{\delta R} |V(\delta)| \]

\[ \leq c \frac{x}{R} (\log x)^{-1/(D-1)} \sum_{\delta > U_x} \frac{|V(\delta)|}{\delta} \]

\[ + \frac{x}{R} \sum_{\sqrt{x} \leq \delta \leq x/R} \frac{|V(\delta)|}{\delta}. \]

(5.10)

Now, on the one hand,

\[ \sum_{\delta > U_x} \frac{|V(\delta)|}{\delta} \leq \sum_{\delta | R} \frac{|\mu(\delta_2)|}{\delta_2} \sum_{\delta_1 > U_x/R} \frac{1}{\delta_1^2} \prod_{p | R} \left(1 + \frac{1}{p}\right) \cdot c \frac{\sqrt{R}}{U_x}, \]

(5.11)

while on the other hand,

\[ \sum_{\sqrt{x} \leq \delta \leq x/R} \frac{|V(\delta)|}{\delta} < \sum_{\delta \geq \sqrt{x}} \frac{|V(\delta)|}{\delta} \leq \prod_{p | R} \left(1 + \frac{1}{p}\right) \cdot \left(\frac{R}{\sqrt{x}}\right)^{1/2}. \]

(5.12)

Gathering (5.11) and (5.12) in (5.10), we obtain

\[ \Sigma_2 = o(x(\log x)^{-1/(D-1)}) \quad (x \to \infty). \]

(5.13)

We now consider an estimate for \( M_\delta \) when \( \delta \leq U_x \). Recall that

\[ M_\delta = \sum_{m_0 + t_0R + \delta Rk \leq x/K} u(m_0 + t_0R + \delta Rk). \]

Let \( A = m_0 + t_0R \) and \( B = \delta R \). One can see that \( (A, B) = 1 \). Indeed, first observe that \( (A, R) = 1 \), since in light of (5.7), we have \( (m_0, R) = 1 \). Now, it follows from (5.7) that \( K(m_0 + t_0R) - R(\nu_0 + t_0K) = 1 \). But \( \delta | \nu_0 + t_0K \) implies that \( (m_0 + t_0R, \delta) = 1 \). Therefore, it follows from these observations that \( (A, B) = 1 \).
Thus, with the above notations, $M_\delta$ can be written as

$$M_\delta = \sum_{A+Bk \leq x/K} u(A+Bk) = \frac{1}{\varphi(B)} \sum_{\chi \pmod{B}} \chi(A) \cdot \sum_{n \leq x/K} \chi(n)u(n)$$

$$= M_\delta^{(1)} + M_\delta^{(2)},$$

say. These last two expressions can be written as

$$M_\delta^{(1)} = \frac{1}{\varphi(B)} \sum_{n \leq x/K} \chi_B^{(0)}(n)u(n),$$

$$M_\delta^{(2)} = \frac{1}{\varphi(B)} \sum_{\chi \neq \chi_0} \chi(A) \cdot \sum_{n \leq x/K} \chi(n)u(n).$$

Let $\chi_B \neq \chi_B^{(0)}$. Then $u(p)\chi_B(p) \neq u(p)$ holds for at least one prime $p = p^*$. But then $u(p) = 1$ for every prime $p \equiv p^* \pmod{B}$ and therefore

$$\frac{1}{\varphi(B)} \sum_{p \leq x} u(p)\chi_B(p) \to \tau = \tau_{\chi_B} \quad (x \to \infty),$$

with $\text{Re}(\tau_{\chi_B}) < \tau_{\chi_0}$. Hence, it follows from Theorem B that

$$M_\delta^{(2)} = o\left(\frac{x}{\log x} m(x)\right) \quad (x \to \infty).$$

On the other hand,

$$\sum_{n \leq x/K} u(n)\chi_B^{(0)}(n) = \frac{e^{-\gamma \tau}}{\Gamma(\tau)} \frac{x}{K \log(x/K)} \prod_{p \leq x/K} \left(1 + \frac{u(p)\chi_B^{(0)}(p)}{p}\right)$$

$$+ o\left(\frac{x}{\log x} m(x)\right),$$

while, for $\delta \leq U(x)$, we have $\log(x/K) = (1 + o(1)) \log x$ as $x \to \infty$.

Now,

$$\prod_{p \leq x/K} \left(1 + \frac{u(p)\chi_B^{(0)}(p)}{p}\right) = (1 + o(1)) \prod_{p \leq x} \left(1 + \frac{F(p)}{p}\right) \prod_{p|KR \delta} \left(1 + \frac{F(p)}{p}\right)^{-1}.$$

Thus, in light of estimates (5.13) through (5.18), (5.9) becomes

$$S_{K,R}(x) = \frac{e^{-\gamma \tau}}{\Gamma(\tau)} H(K,R)m(x) + o\left(\frac{x}{\log x} m(x)\right),$$

(5.19)
where

\[
H(K, R) = \sum_{(\delta, K)=1} \frac{V(\delta)}{K\varphi(R\delta)} \prod_{p|RK\delta} \left(1 + \frac{F(p)}{p}\right)^{-1}
\]

\[
= \prod_{p|RK} \left(1 + \frac{F(p)}{p}\right)^{-1} \sum_{\delta_2|R} \frac{\mu(\delta_2)}{\delta_2} \frac{1}{K\varphi(R)}
\times \sum_{(\delta_2, RK)=1} \frac{\mu(\delta_2)}{\varphi(\delta_2^2)} \prod_{p|\delta_2} \left(1 + \frac{F(p)}{p}\right)^{-1}.
\]

(5.20)

Since \( \sum_{\delta_2|R} \frac{\mu(\delta_2)}{\delta_2} = \frac{\varphi(R)}{R} \) and

\[
\sum_{(\delta_2, RK)=1} \frac{\mu(\delta_2)}{\varphi(\delta_2^2)} \prod_{p|\delta_2} \left(1 + \frac{F(p)}{p}\right)^{-1} = \prod_{p|RK} \left(1 - \frac{1}{p(p-1)} \cdot \frac{1}{1 + \frac{F(p)}{p}}\right),
\]

it follows that (5.20) can be written as

\[
H(K, R) = \frac{1}{KR} \prod_{p|RK} \left(1 + \frac{F(p)}{p}\right)^{-1} \cdot \prod_{p|RK} \left(1 - \frac{1}{p(p-1)} \cdot \frac{1}{1 + \frac{F(p)}{p}}\right).
\]

(5.21)

Note that here we used the fact that

\[
\sum_{\delta \leq U_x} \frac{V(\delta)}{K\varphi(R\delta)} \prod_{p|RK\delta} \left(1 + \frac{F(p)}{p}\right)^{-1} \to H(K, R) \quad \text{as} \quad U_x \to \infty
\]

since it is clear from (5.21) that

\[
0 < \sum_{K, R \in \mathcal{F}}^* H(K, R) < +\infty,
\]

where the star in the sum is there to indicate that we have rightfully ignored those pairs \( K, R \) for which either \( R \in \mathcal{R}_1 \) or \( K, R \) is a non-admissible pair. The statement of Theorem 2.2 then follows from relation (5.19).

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