ON STRINGS OF CONSECUTIVE INTEGERS
WITH A DISTINCT NUMBER OF PRIME FACTORS

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Abstract. Let \( \omega(n) \) be the number of distinct prime factors of \( n \). For any positive integer \( k \) let \( n = n_k \) be the smallest positive integer such that \( \omega(n+1), \ldots, \omega(n+k) \) are mutually distinct. We study the same quantity when \( \omega(n) \) is replaced by \( \Omega(n) \), the total number of prime factors of \( n \) counted with repetitions.

Let \( \omega(n) \) and \( \Omega(n) \) denote respectively the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \) counted with repetitions. For any positive integer \( k \) let \( n = n_k \) be the smallest positive integer such that \( \Omega(n+1), \ldots, \Omega(n+k) \) are mutually distinct. We also let \( m = m_k \) be the smallest positive integer \( m \) such that \( \Omega(m+1), \ldots, \Omega(m+k) \) are mutually distinct. Using a computer, we easily obtain that

\[
\begin{align*}
n_2 & = 4, \\
n_3 & = 27, \\
n_4 & = 416, \\
n_5 & = 14321, \\
n_6 & = 461889, \\
n_7 & = 7362724274, \\
n_8 & = 189374, \\
m_9 & = 46908263 \\
m_{10} & = 683441870.
\end{align*}
\]

In this paper, we give upper and lower bounds for \( n_k \) and \( m_k \). Let \( p_i \) be the \( i \)-th prime number. Let \( n = n_k \). Since the set \( \{\omega(n+j) : j = 1, \ldots, k\} \) consists of \( k \) nonnegative integers, it follows that one of \( n+j \) for \( j = 1, \ldots, k \) must have at least \( k \) distinct prime factors. Thus,

\[
n + k \geq \prod_{i=1}^{k} p_i = \exp((1 + o(1))k) = \exp((1 + o(1))k \log k)
\]

as \( k \to \infty \) by the Prime Number Theorem; therefore

\[
n_k \geq \exp((1 + o(1))k \log k) \quad \text{as} \quad k \to \infty.
\]

Similarly, letting \( m = m_k \), we get that \( \Omega(m+i) \geq k \) for some \( i \in \{1, \ldots, k\} \). Thus,

\[
m + k \geq 2^k
\]

and letting \( m = m_k \), we get that \( \exp((1 + o(1))k \log k) \) as \( k \to \infty \).

We start by improving these trivial estimates as follows.

**Theorem 1.** The inequality

\[
n_k \geq \exp((2 + o(1))k \log k)
\]

holds as \( k \to \infty \). Furthermore, the inequality

\[
m_k \geq \exp((1/2 + o(1))k \log k)
\]

holds as \( k \to \infty \).
The problem of finding lower and upper bounds for \( n_k \) and \( m_k \) was raised in the recent book [1] by the first author. We remark that, after writing this paper, we noticed that the first of these bounds is essentially equivalent to one due to Erdős [2]. We were somewhat surprised that we could not find any other work on these problems.

Proof. We start with the first inequality. Assume that \( \omega(n+1), \ldots, \omega(n+k) \) are mutually distinct. Let \( \varepsilon \in (0,1) \) be arbitrarily small but fixed. Put \( s = [k^{1-\varepsilon}] \). Let \( i_1, \ldots, i_s \) be \( s \) distinct integers in \( \{1, \ldots, k\} \) such that \( \omega(n+i_j) \geq k-j \) for \( j = 1, \ldots, s \). Let \( \mathcal{A}_{ij} \) be the set of prime factors of \( n+i_j \). Note that if \( j \neq \ell \) and \( p \in \mathcal{A}_{ij} \cap \mathcal{A}_{i\ell} \), then \( p \mid (n+i_j) - (n+i_\ell) = (i_j - i_\ell) \) and \( 1 \leq |i_j - i_\ell| \leq k-1 \). Since

\[
\omega(m) \ll \log m / \log \log m
\]

holds for all positive integers, we get that

\[
\#(\mathcal{A}_{ij} \cap \mathcal{A}_{i\ell}) < c_1 \frac{\log k}{\log \log k}
\]

holds for all \( j \neq \ell \) with some absolute constant \( c_1 \). By the Principle of Inclusion and Exclusion,

\[
\# \left( \bigcup_{j=1}^{s} \mathcal{A}_{ij} \right) \geq \sum_{j=1}^{s} \# \mathcal{A}_{ij} - \sum_{1 \leq j < \ell \leq s} \#(\mathcal{A}_{ij} \cap \mathcal{A}_{i\ell}) \geq ks - \frac{s(s+1)}{2} - c_1 \frac{s}{2} \frac{\log k}{\log \log k} > (1-\varepsilon)k^{2-\varepsilon}
\]

provided that \( k > k_\varepsilon \). Thus, using the Prime Number Theorem once more, we have

\[
(n+k)^s \geq \prod_{j=1}^{s} (n+i_j) \geq \prod_{1 \leq i < (1-\varepsilon)k^{2-\varepsilon}} p_i \geq \exp \left( (2-\varepsilon + o(1))ks \log k \right)
\]

as \( k \to \infty \). This leads to \( n \geq \exp((2-\varepsilon + o(1))k \log k) \) as \( k \to \infty \), which implies the desired conclusion since \( \varepsilon \in (0,1) \) was arbitrary.

We now deal with the second inequality. Let \( m = m_k \). For any given prime number \( p \) and positive integer \( n \) we let \( \nu_p(n) \) be the exact exponent with which \( p \) appears in the prime factorization of \( n \). For each \( p \leq k \) let \( i_p \in \{1, \ldots, k\} \) be such that

\[
(\nu_p(m+i_p) = \max_{1 \leq i \leq k} \nu_p(m+i).
\]

If more than one value for \( i_p \in \{1, \ldots, k\} \) exists for which equality (1) is satisfied, we simply pick one of them. Clearly, the set \( \mathcal{I} \) of indices \( i_p \) so chosen satisfies

\[
\# \mathcal{I} \leq \pi(k).
\]

An elementary argument (see, for example, Lemma 2 in [3]) shows that if we write \( m+i = a_i b_i \), where the largest prime factor of \( a_i \) is \( \leq k \) and the smallest prime factor of \( b_i \) exceeds \( k \), then

\[
\prod_{1 \leq i \leq k \atop i \notin \mathcal{I}} a_i \leq k^k.
\]
In particular,

\[
\sum_{1 \leq i \leq k} \Omega(a_i) = \Omega \left( \prod_{1 \leq i \leq k} a_i \right) < \frac{k \log k}{\log 2} < 2k \log k.
\]

Let

\[ \mathcal{J} = \{ i \notin I : \Omega(a_i) > k^{1/2} \} \]

Then inequality (3) shows that

\[
\# \mathcal{J} < 2k^{1/2} \log k.
\]

Finally, let

\[ \mathcal{K} = \{ i \notin I \cup \mathcal{J} : \Omega(m+i) \leq k^{2/3} \} \]

Since the numbers \( \Omega(m+j) \) are distinct for \( j = 1, \ldots, k \), it follows that

\[
\# \mathcal{K} \leq k^{2/3}.
\]

Let \( \mathcal{S} = \{1, \ldots, k\} - (I \cup \mathcal{J} \cup \mathcal{K}) \) and put \( s = \# \mathcal{S} \). Let \( \varepsilon > 0 \) be fixed. Estimates (2), (4) and (5) show that

\[
s \geq k - \pi(k) - 2k^{1/2} \log k - k^{2/3} > (1 - \varepsilon)k,
\]

provided that \( k > k_\varepsilon \). Note that if \( i \in \mathcal{S} \), then

\[
\Omega(a_i) \leq k^{1/2} = (k^{2/3})^{3/4} \leq \Omega(m+i)^{3/4},
\]

so that

\[
\Omega(b_i) = \Omega(m+i) - \Omega(a_i) \geq \Omega(m+i) - \Omega(m+i)^{3/4} \geq (1 - \varepsilon)\Omega(m+i)
\]

for all \( i \in \mathcal{S} \), assuming that \( k > k_\varepsilon \). Thus, since the \( \Omega(m+i) \) are distinct,

\[
(m+k)^s \geq \prod_{i \in \mathcal{S}} b_i > k^{\sum_{i \in \mathcal{S}} \Omega(b_i)} > \left( k^{\sum_{i \in \mathcal{S}} \Omega(m+i)} \right)^{(1-\varepsilon)}
\]

\[
> \left( k^{\sum_{j=1}^{m} j} \right)^{(1-\varepsilon)} > \exp((1/2 - \varepsilon)s^2 \log k).
\]

Hence,

\[
m_k \geq \exp((1/2 - \varepsilon)s \log k) > \exp((1/2 - 2\varepsilon)k \log k).
\]

Since \( \varepsilon > 0 \) is arbitrary, we get the desired conclusion. \( \square \)

We next turn our attention to upper bounds for \( n_k \) and \( m_k \). We have the following result.

**Theorem 2.** The inequalities

\[
n_k \leq \exp((6/ \log 2 + o(1))k^2(\log k)^2)
\]

and

\[
m_k \leq \exp((4/ \log 2 + o(1))k^2(\log k)^2)
\]

hold as \( k \to \infty \).
Proof: We assume that $k \geq 2$. Again, we deal first with $n_k$. We let $A$ be a positive integer depending on $k$, to be determined later. We let $q_1 < q_2 < \cdots < q_m < \cdots$ be all the consecutive prime numbers exceeding $k$. For $j = 1, \ldots, k$, we put $T_j = j(j-1)/2$ and

\[ M_j = \prod_{\ell = T_j+1}^{T_{j+1} - 1} q_\ell. \]

Put $M = \prod_{j=1}^k M_j$ and let $N$ be the smallest positive integer such that $M_j$ divides $N + j$ for each $j$ with $1 \leq j \leq k$. Such an integer $N$ exists by the Chinese Remainder Theorem. Note that $N + k < M$. Indeed, if not, then $N = M - i$ for some $i \in \{1, \ldots, k\}$, and by taking some $j \neq i \in \{1, \ldots, k\}$ (which exists because $k \geq 2$), we would get that $M_j \mid N + j = M + (j - i)$; therefore $M_j \mid j - i$, which is impossible. Let $n = M\lambda + N$ be a positive integer with $\lambda \in [M, 2M]$. Note that

\[ n + j = M\lambda + (N + j) = M_j ((M/M_j)\lambda + (N + j)/M_j), \quad j = 1, \ldots, k. \]

By setting $A_j = (N + j)/M_j$ and $B_j = M/M_j$, it follows that

\[ jA = T_{j+1}A - T_jA = \omega(M_j) \leq \omega(n + j) \leq jA + \omega(B_j\lambda + A_j), \]

so that if $\lambda$ is such that

\[ \omega(B_j\lambda + A_j) < A, \quad \text{for all } j = 1, \ldots, k-1, \]

then

\[ jA \leq \omega(n + j) < jA + A \leq \omega(n + j + 1) \quad \text{for all } j = 1, \ldots, k-1. \]

Hence, we certainly have that $\omega(n+1), \ldots, \omega(n+k)$ are pairwise distinct.

It now remains to estimate $A$ and $M$ such that we can guarantee the existence of a positive integer $\lambda \in [M, 2M]$ with the property that all of the inequalities hold.

We claim that $A_j$ and $B_j$ are coprime. Indeed, to see this, note first that

\[ B_j = M/M_j = \prod_{1 \leq \ell \leq k \atop \ell \neq j} M_\ell. \]

If there exists a prime $p \mid (A_j, B_j)$, we then get that $p \mid M_\ell$ for some $\ell \neq j$. Since $M_\ell \mid N + \ell$, we get that $p \mid N + \ell$. But obviously $p \mid A_j \mid N + j$; therefore $p \mid (N + \ell) - (N + j) = (\ell - j)$, and $1 \leq |\ell - j| < k$. Thus $p < k$, which is impossible because all prime factors of $M$ exceed $k$, proving the claim.

Now note that since $N + k \leq M$, we have

\[ B_j\lambda + A_j \leq \frac{1}{M_j} (M\lambda + N + k) < \frac{2M\lambda}{M_j} \leq \frac{4M^2}{M_j} < M^2 \]

for all $\lambda \in [M, 2M]$ and $j = 1, \ldots, k$, when $k \geq 3$, because in this case all primes dividing $M$ exceed 4 and $N + k < M$. Thus, writing $\tau(m)$ for the number of divisors of $m$, we obtain

\[ \tau(B_j\lambda + A_j) \leq 2 \sum_{d \mid B_j\lambda + A_j \atop d \leq M} 1. \]
Summing the above inequality over all $\lambda \in [M,2M]$ and changing the order of summation, we find that
\[
\sum_{\lambda \in [M,2M]} \tau(B_j\lambda + A_j) \leq 2 \sum_{\lambda \in [M,2M]} \sum_{d|B_j\lambda + A_j} 1 \leq 2 \sum_{d \leq M} \sum_{\lambda \in [M,2M] \ (\text{mod} \ d)} 1 \\
\leq 2 \sum_{d \leq M} \left(\left\lfloor \frac{M}{d}\right\rfloor + 1\right) \leq 4M \sum_{d \leq M} \frac{1}{d} \\
\leq 4M(\log M + 1).
\]
(7)

In the above chain of inequalities, we used the fact that, since $A_j$ and $B_j$ are coprime, the congruence $B_j\lambda + A_j \equiv 0 \ (\text{mod} \ d)$ has at most $\lfloor M/d \rfloor$ solutions $\lambda \in [M,2M]$. This is true assuming that $d$ and $B_j$ are coprime. When $d$ and $B_j$ are not coprime, then this congruence has no integer solution $\lambda$. Thus, if $\lambda$ is such that $\omega(B_j\lambda + A_j) \geq A$, then $\tau(B_j\lambda + A_j) \geq 2^A$ and inequality (7) shows that
\[
\#\{\lambda \in [M,2M] : \omega(B_j\lambda + A_j) \geq A\} \leq \frac{4M(\log M + 1)}{2^A}.
\]

Summing the above inequality over $j = 1, \ldots, k - 1$, we get that
\[
\sum_{j=1}^{k-1} \#\{\lambda \in [M,2M] : \omega(B_j\lambda + A_j) \geq A\} \leq \frac{4(k-1)M(\log M + 1)}{2^A}.
\]

Hence, assuming that
\[
M > \frac{4(k-1)M(\log M + 1)}{2^A},
\]
we see that there exists a number $\lambda \in [M,2M]$ such that all inequalities (6) are satisfied, and therefore
\[
n < n + 1 = M\lambda + N + 1 < 2M^2 + M < 3M^2.
\]

It remains to estimate the size of the minimal integer $A$ depending on $k$ such that inequality (8) holds. Clearly, $M$ has $Ak(k+1)/2$ prime factors, which are all the consecutive primes starting with the first one exceeding $k$. Thus, by the Prime Number Theorem,
\[
M = \exp((1/2 + o(1))k^2A(\log k^2A))
\]
as $k \to \infty$ uniformly in $A \geq 1$. Thus, inequality (8) is fulfilled when
\[
A \log 2 > \log(4(k-1)) + \log(\log M + 1) = (3 + o(1)) \log k + O(\log \log k + \log A).
\]
This shows that given $\varepsilon > 0$, we may choose $A = \lfloor (3/\log 2 + \varepsilon) \log k \rfloor$, and then inequality (8) is fulfilled once $k > k_\varepsilon$. With this choice of $A$, we have that
\[
M < \exp((3/\log 2 + 2\varepsilon)k^2(\log k)^2)
\]
provided that $k$ is sufficiently large, and now inequality (9) shows that
\[
n < \exp((6/\log 2 + 5\varepsilon)k^2(\log k)^2)
\]
if $k$ is sufficiently large with respect to $\varepsilon$, which implies the desired estimate as $k \to \infty$, since $\varepsilon \in (0,1)$ may be chosen arbitrarily small.

We now turn our attention to the upper bound for $m_k$. We follow the same line of attack, based on the Chinese Remainder Theorem, although the details are somewhat different.
We assume again that \( k \geq 2 \); we take \( M_0 = (k!)^2 \), \( M_j = q_j^{i^A} \) for \( j = 1, \ldots, k \), and let \( N \) be the smallest positive integer \( m \) in the arithmetic progression

\[ m + j \equiv 0 \pmod{M_j}, \quad j = 0, \ldots, k. \]

Here,

\[ M = \prod_{j=0}^{k} M_j = (k!)^2 \prod_{j=1}^{k} q_j^{i^A} = \exp((1/2 + o(1))k^2A \log k) \]

as \( k \to \infty \). Let \( m = M\lambda + N \) again be such that \( \lambda \in [M, 2M] \). Then

\[ m + i = iM_i \left( \frac{M}{iM_i} + \frac{N + i}{iM_i} \right), \quad \text{for all } i = 1, \ldots, k, \]

so that if we set \( A_i = (N + i)/(iM_i) \) and \( B_i = M/(iM_i) \), we have

\[ \Omega(m + i) = \Omega(i) + \Omega(M_i) + \Omega(B_i\lambda + A_i). \]

Now since \( i \leq k \), it follows that \( \Omega(i) \leq (\log k)/\log 2 \). Furthermore, \( \Omega(M_i) = iA \). Thus, if

\[ \Omega(B_i\lambda + A_i) < A - (\log k)/\log 2, \quad \text{for all } i = 1, \ldots, k - 1, \]

then

\[ \Omega(m + i) < A(i + 1) = \Omega(M_{i+1}) < \Omega(m + i + 1), \quad \text{for all } i = 1, \ldots, k - 1, \]

which certainly shows that \( \Omega(m + 1), \ldots, \Omega(m + k) \) are pairwise distinct.

Now let \( i \in \{1, \ldots, k\} \). As in the analysis of the \( n_k \) case, one shows that \( A_i \) and \( B_i \) are coprime and that \( B_i\lambda + A_i < M\lambda + N + k < 2M^2 + M < 3M^2 \). Furthermore, since \( M_0/i \) is a divisor of \( B_i \) for all \( i = 1, \ldots, k \) and \( M_0/i = (k!)^2/i \) is divisible by all primes \( p \leq k \), it follows that the smallest prime factor of \( B_i\lambda + A_i \) exceeds \( k \).

Write

\[ B_i\lambda + A_i = U_i V_i, \]

where all prime factors of \( U_i \) are \( \leq M^{1/2} \) and all prime factors of \( V_i \) are \( > M^{1/2} \). Clearly, \( \Omega(V_i) \leq 4 \) because \( M > 9 \). We will now bound from above the number of \( \lambda \) such that \( U_i \) is not squarefree for some \( i = 1, \ldots, k \). There exists a prime \( p \in [k, M^{1/2}] \) such that \( B_i\lambda + A_i \equiv 0 \pmod{p^2} \). For a fixed prime \( p \), the number of integers \( \lambda \in [M, 2M] \) for which the above congruence holds is at most \( [M/p^2] + 1 \leq 2M/p^2 \). Thus,

\[ \#\{\lambda \in [M, 2M] : p^2 \mid B_i\lambda + A_i \text{ for some } p \in [k, M^{1/2}]\} \leq 2M \sum_{p>k} \frac{1}{p^2} \leq \frac{M}{k \log k} \]

uniformly in \( i \in \{1, \ldots, k\} \). Summing this over all \( i \in \{1, \ldots, k\} \), we get that

\[ \sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : U_i \text{ is not squarefree}\} \ll \frac{M}{\log k}. \]

In particular, if \( k \) is large, then

\[ \sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : \Omega(B_i\lambda + A_i) > \omega(B_i\lambda + A_i) + 4\} \ll \frac{M}{2}. \]
Let \(\lambda\) be some number in \([M, 2M]\) such that \(\Omega(B_i\lambda + A_i) \leq \omega(B_i\lambda + A_i) + 4\). As we have seen, there are at least \(M/2\) such values for \(\lambda\). If there is such a positive integer \(\lambda\) with the additional property that

\[
\omega(B_i\lambda + A_i) < A - \frac{\log k}{\log 2} - 4,
\]

for all \(i = 1, \ldots, k\), it follows that inequalities (11) are satisfied. So, let us look at the number of \(\lambda \in [M, 2M]\) such that at least one of the inequalities (12) fails. The argument used in the proof of the upper bound for \(n_k\) (based on the fact that \(\tau(m) \geq 2^{\omega(m)}\)) shows that

\[
\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : \omega(B_i\lambda + A_i) \geq A - (\log k)/\log 2 - 4\} \leq 4(k-1)M(\log M) \quad \frac{2A-(\log k)/(\log 2)-4}{2A-(\log k)/(\log 2)-4}.
\]

Thus, if

\[
4(k-1)M(\log M) < \frac{M}{2},
\]

then the number of \(\lambda \in [M, 2M]\) such that at least one of the inequalities (12) fails is < \(M/2\). Since we have \(\geq M/2\) values of \(\lambda\) to choose from, it follows that one can indeed choose such a value of \(\lambda\) for which all inequalities in (11) hold. Clearly, with such a value of \(\lambda\), we have that \(m_k \leq m = M\lambda + N < 3M^2\). Inequality (13) is equivalent, via estimate (10), to

\[
A \log 2 - (\log k) - 4 \log 2 > \log(8(k - 1)) + 2 \log k + O(\log \log k + \log A),
\]

which holds if we first fix \(\varepsilon > 0\), then take \(k > k_\varepsilon\), and finally choose \(A = [(4/\log 2 + \varepsilon) \log k]\). With this choice of \(A\), we have

\[
M < \exp((2/\log 2 + 2\varepsilon)k^2(\log k)^2)
\]

once \(k > k_\varepsilon\). Therefore,

\[
m_k < 3M^2 < \exp((4/\log 2 + 5\varepsilon)k^2(\log k)^2)
\]

if \(k\) is large with respect to \(\varepsilon\), which implies the desired inequality since \(\varepsilon > 0\) can be chosen arbitrarily small. \(\square\)

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References


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